Problem Set 5

Math 205A: Winter Quarter, 2014

1. Suppose that $\Omega \subset \mathbb{C}$ is a simply connected open set, $\gamma : [a, b] \to \Omega$ is a smooth curve in Ω, and $c \in \mathbb{C} \setminus \Omega$. Show that $W_\gamma(c) = 0$.

2. A linear fractional transformation $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ on the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined for $ad - bc \neq 0$ by

$$f(z) = \frac{az + b}{cz + d},$$

with $f(\infty) = a/c$ and $f(-d/c) = \infty$ if $c \neq 0$, or $f(\infty) = \infty$ if $c = 0$.

(a) Verify that a linear fractional transformation is a biholomorphic mapping of $\bar{\mathbb{C}}$ onto itself.

(b) Show that every biholomorphic mapping of $\bar{\mathbb{C}}$ onto itself is a linear fractional transformation. (You can assume the result we proved in class that every holomorphic mapping $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is rational.)

3. Let Ω be an open subset of $\mathbb{C}$ that is star-shaped with respect to $c \in \Omega$ (i.e., for every $z \in \Omega$, the line segment from c to z is included in Ω).

(a) Prove that every closed curve $\gamma : [a, b] \to \Omega \setminus \{c\}$ is homotopic to a closed curve $\beta : [a, b] \to C_r$ where $C_r = \{z \in \mathbb{C} : |z - c| = r\}$ is any circle of sufficiently small radius $r > 0$.

(b) Show that two closed curves $\alpha, \beta : [a, b] \to C_r$ are homotopic if and only if they have the same winding numbers i.e., $W_\alpha(c) = W_\beta(c)$. Deduce that two closed curves are homotopic in $\Omega \setminus \{c\}$ if and only if their winding numbers about c are equal.

(c) If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma : [a, b] \to \Omega \setminus \{c\}$ is a closed curve, use the homotopy form of Cauchy's theorem to prove that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - c}\, dz = W_\gamma(c) f(c).$$

4. Consider the closed curve γ in $\Omega \setminus \{0, 1\}$ shown in Figure 1.

(a) What are the winding numbers $W_\gamma(0)$ and $W_\gamma(1)$?

(b) Is γ homotopic to 0 in $\Omega \setminus \{0, 1\}$?

(c) If f is holomorphic in $\Omega \setminus \{0, 1\}$ with poles at 0 and 1, what is $\int_\gamma f\, dz$? (You should explain your answers, but proofs are not required.)

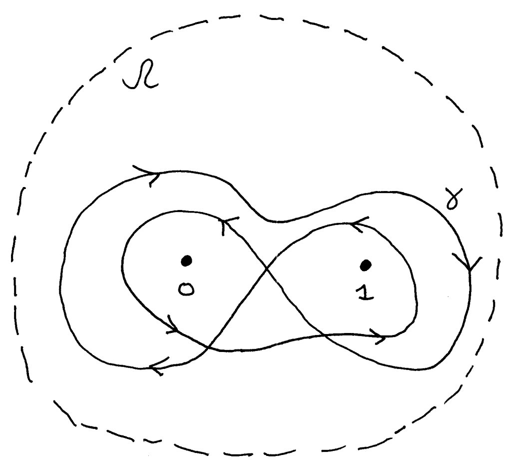

Figure 1: Curve γ for Problem 4.