**Problem Set 3**

Math 205B: Spring Quarter, 2014

1. If $U \subset \mathbb{C}$ is an open set, prove that every two curves in $U$ with the same endpoints are fixed-end-point homotopic if and only if every closed curve is homotopic to a constant curve.

2. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $f(z) = z\sqrt{1+z}$, with the principal branch of the square-root. Describe the Riemann surface that is obtained from analytic continuation of the function element $(f, U)$. How many points in the Riemann surface lie above $z = 0$? Explain your answer.

3. (a) Let $0 < R(a) < \infty$ denote the radius of convergence of a power series at $a \in \mathbb{C}$, and suppose that $|b - a| < \frac{1}{2}R(a)$. Prove that the radius of convergence $R(b)$ of the power series at $b$ of the analytic continuation of the power series at $a$ satisfies $|R(a) - R(b)| \leq |a - b|$. (You can assume standard facts about the convergence of power series.)

(b) Deduce that if a holomorphic germ has an analytic continuation along a curve $\gamma : [0, 1] \to \mathbb{C}$, then the analytic continuation can always be obtained by analytic continuation along a finite chain of discs.

(c) Show that if $\mathbf{f}$ is the complete analytic continuation of a holomorphic germ $f_a$, then the collection of germs $\{f_z \in \mathbf{f}\}$ is countable for every $z \in \mathbb{C}$. *Hint.* Continue analytically by discs with rational centers.

4. (a) Consider a linear ODE

$$a(z)f'' + b(z)f' + c(z) = 0,$$

where the coefficients $a$, $b$, $c$ are entire functions. Show that if a function element $(f, U)$ satisfies this ODE in $U$, then every analytic continuation $(g, V)$ of $(f, U)$ satisfies the ODE in $V$.

(b) Solve the ODE

$$z^2 f'' - zf' + f = 0,$$

and discuss what happens when solutions are continued analytically around the unit circle $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi it}$.