CHAPTER 1

Measures

Measures are a generalization of volume; the fundamental example is Lebesgue measure on $\mathbb{R}^n$, which we discuss in detail in the next Chapter. Moreover, as formalized by Kolmogorov (1933), measure theory provides the foundation of probability. Measures are important not only because of their intrinsic geometrical and probabilistic significance, but because they allow us to define integrals.

This connection, in fact, goes in both directions: we can define an integral in terms of a measure; or, in the Daniell-Stone approach, we can start with an integral (a linear functional acting on functions) and use it to define a measure. In probability theory, this corresponds to taking the expectation of random variables as the fundamental concept from which the probability of events is derived.

In these notes, we develop the theory of measures first, and then define integrals. This is (arguably) the more concrete and natural approach; it is also (unarguably) the original approach of Lebesgue. We begin, in this Chapter, with some preliminary definitions and terminology related to measures on arbitrary sets. See Folland [**4**] for further discussion.

1.1. Sets

We use standard definitions and notations from set theory and will assume the axiom of choice when needed. The words 'collection' and 'family' are synonymous with 'set' — we use them when talking about sets of sets. We denote the collection of subsets, or power set, of a set X by $\mathcal{P}(X)$. The notation 2^X is also used.

If $E \subset X$ and the set X is understood, we denote the complement of E in X by $E^c = X \setminus E$. De Morgan's laws state that

$$\left(\bigcup_{\alpha\in I} E_\alpha\right)^c = \bigcap_{\alpha\in I} E_\alpha^c, \qquad \left(\bigcap_{\alpha\in I} E_\alpha\right)^c = \bigcup_{\alpha\in I}^{\infty} E_\alpha^c.$$

We say that a collection

$$\mathcal{C} = \{E_\alpha \subset X : \alpha \in I\}$$

of subsets of a set X, indexed by a set I, covers $E \subset X$ if

$$\bigcup_{\alpha\in I} E_\alpha \supset E.$$

The collection $\mathcal{C}$ is disjoint if $E_\alpha \cap E_\beta = \varnothing$ for $\alpha \neq \beta$.

The Cartesian product, or product, of sets X, Y is the collection of all ordered pairs

$$X \times Y = \{(x, y) : x \in X,\, y \in Y\}.$$

1.2. Topological spaces

A topological space is a set equipped with a collection of open subsets that satisfies appropriate conditions.

Definition 1.1. A topological space $(X, \mathcal{T})$ is a set X and a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X, called open sets, such that

(a) $\varnothing, X \in \mathcal{T}$;

(b) if $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$ is an arbitrary collection of open sets, then their union
$$\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$$
is open;

(c) if $\{U_i \in \mathcal{T} : i = 1, 2, \dots, N\}$ is a finite collection of open sets, then their intersection
$$\bigcap_{i=1}^{N} U_i \in \mathcal{T}$$
is open.

The complement of an open set in X is called a closed set, and $\mathcal{T}$ is called a topology on X.

1.3. Extended real numbers

It is convenient to use the extended real numbers
$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$
This allows us, for example, to talk about sets with infinite measure or non-negative functions with infinite integral. The extended real numbers are totally ordered in the obvious way: ∞ is the largest element, $-\infty$ is the smallest element, and real numbers are ordered as in $\mathbb{R}$. Algebraic operations on $\overline{\mathbb{R}}$ are defined when they are unambiguous *e.g.* $\infty + x = \infty$ for every $x \in \overline{\mathbb{R}}$ except $x = -\infty$, but $\infty - \infty$ is undefined.

We define a topology on $\overline{\mathbb{R}}$ in a natural way, making $\overline{\mathbb{R}}$ homeomorphic to a compact interval. For example, the function $\phi : \overline{\mathbb{R}} \to [-1, 1]$ defined by
$$\phi(x) = \begin{cases} 1 & \text{if } x = \infty \\ x/\sqrt{1+x^2} & \text{if } -\infty < x < \infty \\ -1 & \text{if } x = -\infty \end{cases}$$
is a homeomorphism.

A primary reason to use the extended real numbers is that upper and lower bounds always exist. Every subset of $\overline{\mathbb{R}}$ has a supremum (equal to ∞ if the subset contains ∞ or is not bounded from above in $\mathbb{R}$) and infimum (equal to $-\infty$ if the subset contains $-\infty$ or is not bounded from below in $\mathbb{R}$). Every increasing sequence of extended real numbers converges to its supremum, and every decreasing sequence converges to its infimum. Similarly, if $\{a_n\}$ is a sequence of extended real-numbers then
$$\limsup_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \left(\sup_{i \geq n} a_i \right), \qquad \liminf_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \left(\inf_{i \geq n} a_i \right)$$
both exist as extended real numbers.

Every sum $\sum_{i=1}^{\infty} x_i$ with non-negative terms $x_i \geq 0$ converges in $\overline{\mathbb{R}}$ (to ∞ if $x_i = \infty$ for some $i \in \mathbb{N}$ or the series diverges in $\mathbb{R}$), where the sum is defined by

$$\sum_{i=1}^{\infty} x_i = \sup \left\{ \sum_{i \in F} x_i : F \subset \mathbb{N} \text{ is finite} \right\}.$$

As for non-negative sums of real numbers, non-negative sums of extended real numbers are unconditionally convergent (the order of the terms does not matter); we can rearrange sums of non-negative extended real numbers

$$\sum_{i=1}^{\infty} (x_i + y_i) = \sum_{i=1}^{\infty} x_i + \sum_{i=1}^{\infty} y_i;$$

and double sums may be evaluated as iterated single sums

$$\begin{aligned} \sum_{i,j=1}^{\infty} x_{ij} &= \sup \left\{ \sum_{(i,j) \in F} x_{ij} : F \subset \mathbb{N} \times \mathbb{N} \text{ is finite} \right\} \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) \\ &= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} x_{ij} \right). \end{aligned}$$

Our use of extended real numbers is closely tied to the order and monotonicity properties of $\mathbb{R}$. In dealing with complex numbers or elements of a vector space, we will always require that they are strictly finite.

1.4. Outer measures

As stated in the following definition, an outer measure is a monotone, countably subadditive, non-negative, extended real-valued function defined on all subsets of a set.

Definition 1.2. An outer measure μ^* on a set X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that:

(a) $\mu^*(\varnothing) = 0$;
(b) if $E \subset F \subset X$, then $\mu^*(E) \leq \mu^*(F)$;
(c) if $\{E_i \subset X : i \in \mathbb{N}\}$ is a countable collection of subsets of X, then

$$\mu^* \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

We obtain a statement about finite unions from a statement about infinite unions by taking all but finitely many sets in the union equal to the empty set. Note that μ^* is not assumed to be additive even if the collection $\{E_i\}$ is disjoint.

1.5. σ-algebras

A σ-algebra on a set X is a collection of subsets of a set X that contains $\varnothing$ and X, and is closed under complements, finite unions, countable unions, and countable intersections.

Definition 1.3. A σ-algebra on a set X is a collection $\mathcal{A}$ of subsets of X such that:
(a) $\varnothing, X \in \mathcal{A}$;
(b) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;
(c) if $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$ then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}, \qquad \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}.$$

From de Morgan's laws, a collection of subsets is σ-algebra if it contains $\varnothing$ and is closed under the operations of taking complements and countable unions (or, equivalently, countable intersections).

Example 1.4. If X is a set, then $\{\varnothing, X\}$ and $\mathcal{P}(X)$ are σ-algebras on X; they are the smallest and largest σ-algebras on X, respectively.

Measurable spaces provide the domain of measures, defined below.

Definition 1.5. A measurable space $(X, \mathcal{A})$ is a non-empty set X equipped with a σ-algebra $\mathcal{A}$ on X.

It is useful to compare the definition of a σ-algebra with that of a topology in Definition 1.1. There are two significant differences. First, the complement of a measurable set is measurable, but the complement of an open set is not, in general, open, excluding special cases such as the discrete topology $\mathcal{T} = \mathcal{P}(X)$. Second, countable intersections and unions of measurable sets are measurable, but only finite intersections of open sets are open while arbitrary (even uncountable) unions of open sets are open. Despite the formal similarities, the properties of measurable and open sets are very different, and they do not combine in a straightforward way.

If $\mathcal{F}$ is any collection of subsets of a set X, then there is a smallest σ-algebra on X that contains $\mathcal{F}$, denoted by $\sigma(\mathcal{F})$.

Definition 1.6. If $\mathcal{F}$ is any collection of subsets of a set X, then the σ-algebra generated by $\mathcal{F}$ is

$$\sigma(\mathcal{F}) = \bigcap \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \supset \mathcal{F} \text{ and } \mathcal{A} \text{ is a } \sigma\text{-algebra}\}.$$

This intersection is nonempty, since $\mathcal{P}(X)$ is a σ-algebra that contains $\mathcal{F}$, and an intersection of σ-algebras is a σ-algebra. An immediate consequence of the definition is the following result, which we will use repeatedly.

Proposition 1.7. *If $\mathcal{F}$ is a collection of subsets of a set X such that $\mathcal{F} \subset \mathcal{A}$ where $\mathcal{A}$ is a σ-algebra on X, then $\sigma(\mathcal{F}) \subset \mathcal{A}$.*

Among the most important σ-algebras are the Borel σ-algebras on topological spaces.

Definition 1.8. Let $(X, \mathcal{T})$ be a topological space. The Borel σ-algebra

$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

is the σ-algebra generated by the collection $\mathcal{T}$ of open sets on X.

1.6. Measures

A measure is a countably additive, non-negative, extended real-valued function defined on a σ-algebra.

Definition 1.9. A measure μ on a measurable space $(X, \mathcal{A})$ is a function

$$\mu : \mathcal{A} \to [0, \infty]$$

such that

(a) $\mu(\varnothing) = 0$;
(b) if $\{A_i \in \mathcal{A} : i \in \mathbb{N}\}$ is a countable disjoint collection of sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

In comparison with an outer measure, a measure need not be defined on all subsets of a set, but it is countably additive rather than countably subadditive. A measure μ on a set X is finite if $\mu(X) < \infty$, and σ-finite if $X = \bigcup_{n=1}^{\infty} A_n$ is a countable union of measurable sets A_n with finite measure, $\mu(A_n) < \infty$. A probability measure is a finite measure with $\mu(X) = 1$.

A measure space $(X, \mathcal{A}, \mu)$ consists of a set X, a σ-algebra $\mathcal{A}$ on X, and a measure μ defined on $\mathcal{A}$. When $\mathcal{A}$ and μ are clear from the context, we will refer to the measure space X. We define subspaces of measure spaces in the natural way.

Definition 1.10. If $(X, \mathcal{A}, \mu)$ is a measure space and $E \subset X$ is a measurable subset, then the measure subspace $(E, \mathcal{A}|_E, \mu|_E)$ is defined by restricting μ to E:

$$\mathcal{A}|_E = \{A \cap E : A \in \mathcal{A}\}, \qquad \mu|_E\,(A \cap E) = \mu(A \cap E).$$

As we will see, the construction of nontrivial measures, such as Lebesgue measure, requires considerable effort. Nevertheless, there is at least one useful example of a measure that is simple to define.

Example 1.11. Let X be an arbitrary non-empty set. Define $\nu : \mathcal{P}(X) \to [0, \infty]$ by

$$\nu(E) = \text{number of elements in E},$$

where $\nu(\varnothing) = 0$ and $\nu(E) = \infty$ if E is not finite. Then ν is a measure, called counting measure on X. Every subset of X is measurable with respect to ν. Counting measure is finite if X is finite and σ-finite if X is countable.

A useful implication of the countable additivity of a measure is the following monotonicity result.

Proposition 1.12. *If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence of measurable sets, meaning that $A_{i+1} \supset A_i$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i). \tag{1.1}$$

If $\{A_i : i \in \mathbb{N}\}$ is a decreasing sequence of measurable sets, meaning that $A_{i+1} \subset A_i$, and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i). \tag{1.2}$$

Proof. If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence of sets and $B_i = A_{i+1} \setminus A_i$, then $\{B_i : i \in \mathbb{N}\}$ is a disjoint sequence with the same union, so by the countable additivity of μ

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu\left(B_i\right).$$

Moreover, since $A_j = \bigcup_{i=1}^{j} B_i$,

$$\mu(A_j) = \sum_{i=1}^{j} \mu\left(B_i\right),$$

which implies that

$$\sum_{i=1}^{\infty} \mu\left(B_i\right) = \lim_{j\to\infty} \mu(A_j)$$

and the first result follows.

If $\mu(A_1) < \infty$ and $\{A_i\}$ is decreasing, then $\{B_i = A_1 \setminus A_i\}$ is increasing and

$$\mu(B_i) = \mu(A_1) - \mu(A_i).$$

It follows from the previous result that

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i) = \mu(A_1) - \lim_{i\to\infty} \mu(A_i).$$

Since

$$\bigcup_{i=1}^{\infty} B_i = A_1 \setminus \bigcap_{i=1}^{\infty} A_i, \qquad \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1) - \mu\left(\bigcap_{i=1}^{\infty} A_i\right),$$

the result follows. □

Example 1.13. To illustrate the necessity of the condition $\mu(A_1) < \infty$ in the second part of the previous proposition, or more generally $\mu(A_n) < \infty$ for some $n \in \mathbb{N}$, consider counting measure $\nu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ on $\mathbb{N}$. If

$$A_n = \{k \in \mathbb{N} : k \geq n\},$$

then $\nu(A_n) = \infty$ for every $n \in \mathbb{N}$, so $\nu(A_n) \to \infty$ as $n \to \infty$, but

$$\bigcap_{n=1}^{\infty} A_n = \varnothing, \qquad \nu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0.$$

1.7. Sets of measure zero

A set of measure zero, or a null set, is a measurable set N such that $\mu(N) = 0$. A property which holds for all $x \in X \setminus N$ where N is a set of measure zero is said to hold almost everywhere, or a.e. for short. If we want to emphasize the measure, we say μ-a.e. In general, a subset of a set of measure zero need not be measurable, but if it is, it must have measure zero.

It is frequently convenient to use measure spaces which are complete in the following sense. (This is, of course, a different sense of 'complete' than the one used in talking about complete metric spaces.)

Definition 1.14. A measure space $(X, \mathcal{A}, \mu)$ is complete if every subset of a set of measure zero is measurable.

Note that completeness depends on the measure μ, not just the σ-algebra $\mathcal{A}$. Any measure space $(X, \mathcal{A}, \mu)$ is contained in a uniquely defined completion $(X, \overline{\mathcal{A}}, \overline{\mu})$, which the smallest complete measure space that contains it and is given explicitly as follows.

Theorem 1.15. *If $(X, \mathcal{A}, \mu)$ is a measure space, define $(X, \overline{\mathcal{A}}, \overline{\mu})$ by*

$$\overline{\mathcal{A}} = \{A \cup M : A \in \mathcal{A},\ M \subset N \text{ where } N \in \mathcal{A} \text{ satisfies } \mu(N) = 0\}$$

with $\overline{\mu}(A \cup M) = \mu(A)$. Then $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space such that $\overline{\mathcal{A}} \supset \mathcal{A}$ and $\overline{\mu}$ is the unique extension of μ to $\overline{\mathcal{A}}$.

PROOF. The collection $\overline{\mathcal{A}}$ is a σ-algebra. It is closed under complementation because, with the notation used in the definition,

$$(A \cup M)^c = A^c \cap M^c, \qquad M^c = N^c \cup (N \setminus M).$$

Therefore

$$(A \cup M)^c = (A^c \cap N^c) \cup (A^c \cap (N \setminus M)) \in \overline{\mathcal{A}},$$

since $A^c \cap N^c \in \mathcal{A}$ and $A^c \cap (N \setminus M) \subset N$. Moreover, $\overline{\mathcal{A}}$ is closed under countable unions because if $A_i \in \mathcal{A}$ and $M_i \subset N_i$ where $\mu(N_i) = 0$ for each $i \in \mathbb{N}$, then

$$\bigcup_{i=1}^{\infty} A_i \cup M_i = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{i=1}^{\infty} M_i\right) \in \overline{\mathcal{A}},$$

since

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}, \qquad \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} N_i, \quad \mu\left(\bigcup_{i=1}^{\infty} N_i\right) = 0.$$

It is straightforward to check that $\overline{\mu}$ is well-defined and is the unique extension of μ to a measure on $\overline{\mathcal{A}}$, and that $(X, \overline{\mathcal{A}}, \overline{\mu})$ is complete. $\square$