CHAPTER 7

L^p spaces

In this Chapter we consider L^p-spaces of functions whose pth powers are integrable. We will not develop the full theory of such spaces here, but consider only those properties that are directly related to measure theory — in particular, density, completeness, and duality results. The fact that spaces of Lebesgue integrable functions are complete, and therefore Banach spaces, is another crucial reason for the success of the Lebesgue integral. The L^p-spaces are perhaps the most useful and important examples of Banach spaces.

7.1. L^p spaces

For definiteness, we consider real-valued functions. Analogous results apply to complex-valued functions.

Definition 7.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \le p < \infty$. The space $L^p(X)$ consists of equivalence classes of measurable functions $f : X \to \mathbb{R}$ such that

$$\int |f|^p \, d\mu < \infty,$$

where two measurable functions are equivalent if they are equal μ-a.e. The L^p-norm of $f \in L^p(X)$ is defined by

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

The notation $L^p(X)$ assumes that the measure μ on X is understood. We say that $f_n \to f$ in L^p if $\|f - f_n\|_{L^p} \to 0$. The reason to regard functions that are equal a.e. as equivalent is so that $\|f\|_{L^p} = 0$ implies that $f = 0$. For example, the characteristic function $\chi_{\mathbb{Q}}$ of the rationals on $\mathbb{R}$ is equivalent to 0 in $L^p(\mathbb{R})$. We will not worry about the distinction between a function and its equivalence class, except when the precise pointwise values of a representative function are significant.

Example 7.2. If $\mathbb{N}$ is equipped with counting measure, then $L^p(\mathbb{N})$ consists of all sequences $\{x_n \in \mathbb{R} : n \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

We write this sequence space as $\ell^p(\mathbb{N})$, with norm

$$\|\{x_n\}\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

The space $L^\infty(X)$ is defined in a slightly different way. First, we introduce the notion of esssential supremum.

Definition 7.3. Let $f : X \to \mathbb{R}$ be a measurable function on a measure space $(X, \mathcal{A}, \mu)$. The essential supremum of f on X is

$$\operatorname*{ess\,sup}_X f = \inf \{a \in \mathbb{R} : \mu\{x \in X : f(x) > a\} = 0\} .$$

Equivalently,

$$\operatorname*{ess\,sup}_X f = \inf \left\{ \sup_X g : g = f \text{ pointwise a.e.} \right\} .$$

Thus, the essential supremum of a function depends only on its μ-a.e. equivalence class. We say that f is essentially bounded on X if

$$\operatorname*{ess\,sup}_X |f| < \infty.$$

Definition 7.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. The space $L^\infty(X)$ consists of pointwise a.e.-equivalence classes of essentially bounded measurable functions $f : X \to \mathbb{R}$ with norm

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_X |f|.$$

In future, we will write

$$\operatorname{ess\,sup} f = \sup f.$$

We rarely want to use the supremum instead of the essential supremum when the two have different values, so this notation should not lead to any confusion.

7.2. Minkowski and Hölder inequalities

We state without proof two fundamental inequalities.

Theorem 7.5 (Minkowski inequality)**.** *If $f, g \in L^p(X)$, where $1 \le p \le \infty$, then $f + g \in L^p(X)$ and*

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|f\|_{L^p} .$$

This inequality means, as stated previously, that $\| \cdot \|_{L^p}$ is a norm on $L^p(X)$ for $1 \le p \le \infty$. If $0 < p < 1$, then the reverse inequality holds

$$\|f\|_{L^p} + \|g\|_{L^p} \le \|f + g\|_{L^p} ,$$

so $\| \cdot \|_{L^p}$ is not a norm in that case. Nevertheless, for $0 < p < 1$ we have

$$|f + g|^p \le |f|^p + |g|^p,$$

so $L^p(X)$ is a linear space in that case also.

To state the second inequality, we define the Hölder conjugate of an exponent.

Definition 7.6. Let $1 \le p \le \infty$. The Hölder conjugate p' of p is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1 \qquad \text{if } 1 < p < \infty,$$

and $1' = \infty$, $\infty' = 1$.

Note that $1 \le p' \le \infty$, and the Hölder conjugate of p' is p.

Theorem 7.7 (Hölder's inequality)**.** *Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $1 \le p \le \infty$. If $f \in L^p(X)$ and $g \in L^{p'}(X)$, then $fg \in L^1(X)$ and*

$$\int |fg| \, d\mu \le \|f\|_{L^p} \|g\|_{L^{p'}} .$$

For $p = p' = 2$, this is the Cauchy-Schwartz inequality.

7.3. Density

Density theorems enable us to prove properties of L^p functions by proving them for functions in a dense subspace and then extending the result by continuity. For general measure spaces, the simple functions are dense in L^p.

Theorem 7.8. *Suppose that $(X, \mathcal{A}, \nu)$ is a measure space and $1 \le p \le \infty$. Then the simple functions that belong to $L^p(X)$ are dense in $L^p(X)$.*

Proof. It is sufficient to prove that we can approximate a positive function $f : X \to [0, \infty)$ by simple functions, since a general function may be decomposed into its positive and negative parts.

First suppose that $f \in L^p(X)$ where $1 \le p < \infty$. Then, from Theorem 3.12, there is an increasing sequence of simple functions $\{\phi_n\}$ such that $\phi_n \uparrow f$ pointwise. These simple functions belong to L^p, and

$$|f - \phi_n|^p \le |f|^p \in L^1(X).$$

Hence, the dominated convergence theorem implies that

$$\int |f - \phi_n|^p \, d\mu \to 0 \qquad \text{as } n \to \infty,$$

which proves the result in this case.

If $f \in L^\infty(X)$, then we may choose a representative of f that is bounded. According to Theorem 3.12, there is a sequence of simple functions that converges uniformly to f, and therefore in $L^\infty(X)$. □

Note that a simple function

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i}$$

belongs to L^p for $1 \le p < \infty$ if and only if $\mu(A_i) < \infty$ for every A_i such that $c_i \neq 0$, meaning that its support has finite measure. On the other hand, every simple function belongs to L^∞.

For suitable measures defined on topological spaces, Theorem 7.8 can be used to prove the density of continuous functions in L^p for $1 \le p < \infty$, as in Theorem 4.27 for Lebesgue measure on $\mathbb{R}^n$. We will not consider extensions of that result to more general measures or topological spaces here.

7.4. Completeness

In proving the completeness of $L^p(X)$, we will use the following Lemma.

Lemma 7.9. *Suppose that X is a measure space and $1 \le p < \infty$. If*

$$\{g_k \in L^p(X) : k \in \mathbb{N}\}$$

is a sequence of L^p-functions such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^p} < \infty,$$

then there exists a function $f \in L^p(X)$ such that

$$\sum_{k=1}^{\infty} g_k = f$$

where the sum converges pointwise a.e. and in L^p*.*

PROOF. Define $h_n, h : X \to [0,\infty]$ by

$$h_n = \sum_{k=1}^{n} |g_k|, \qquad h = \sum_{k=1}^{\infty} |g_k|.$$

Then $\{h_n\}$ is an increasing sequence of functions that converges pointwise to h, so the monotone convergence theorem implies that

$$\int h^p\, d\mu = \lim_{n\to\infty} \int h_n^p\, d\mu.$$

By Minkowski's inequality, we have for each $n \in \mathbb{N}$ that

$$\|h_n\|_{L^p} \le \sum_{k=1}^{n} \|g_k\|_{L^p} \le M$$

where $\sum_{k=1}^{\infty} \|g_k\|_{L^p} = M$. It follows that $h \in L^p(X)$ with $\|h\|_{L^p} \le M$, and in particular that h is finite pointwise a.e. Moreover, the sum $\sum_{k=1}^{\infty} g_k$ is absolutely convergent pointwise a.e., so it converges pointwise a.e. to a function $f \in L^p(X)$ with $|f| \le h$. Since

$$\left| f - \sum_{k=1}^{n} g_k \right|^p \le \left(|f| + \sum_{k=1}^{n} |g_k| \right)^p \le (2h)^p \in L^1(X),$$

the dominated convergence theorem implies that

$$\int \left| f - \sum_{k=1}^{n} g_k \right|^p d\mu \to 0 \qquad \text{as } n \to \infty,$$

meaning that $\sum_{k=1}^{\infty} g_k$ converges to f in L^p. □

The following theorem implies that $L^p(X)$ equipped with the L^p-norm is a Banach space.

Theorem 7.10 (Riesz-Fischer theorem)**.** *If* X *is a measure space and* $1 \le p \le \infty$*, then* $L^p(X)$ *is complete.*

PROOF. First, suppose that $1 \le p < \infty$. If $\{f_k : k \in \mathbb{N}\}$ is a Cauchy sequence in $L^p(X)$, then we can choose a subsequence $\{f_{k_j} : j \in \mathbb{N}\}$ such that

$$\left\| f_{k_{j+1}} - f_{k_j} \right\|_{L^p} \le \frac{1}{2^j}.$$

Writing $g_j = f_{k_{j+1}} - f_{k_j}$, we have

$$\sum_{j=1}^{\infty} \|g_j\|_{L^p} < \infty,$$

so by Lemma 7.9, the sum

$$f_{k_1} + \sum_{j=1}^{\infty} g_j$$

converges pointwise a.e. and in L^p to a function $f \in L^p$. Hence, the limit of the subsequence

$$\lim_{j\to\infty} f_{k_j} = \lim_{j\to\infty}\left(f_{k_1} + \sum_{i=1}^{j-1} g_i\right) = f_{k_1} + \sum_{j=1}^{\infty} g_j = f$$

exists in L^p. Since the original sequence is Cauchy, it follows that

$$\lim_{k\to\infty} f_k = f$$

in L^p. Therefore every Cauchy sequence converges, and $L^p(X)$ is complete when $1 \le p < \infty$.

Second, suppose that $p = \infty$. If $\{f_k\}$ is Cauchy in L^∞, then for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that we have

$$|f_j(x) - f_k(x)| < \frac{1}{m} \qquad \text{for all } j,k \ge n \text{ and } x \in N^c_{j,k,m} \tag{7.1}$$

where $N_{j,k,m}$ is a null set. Let

$$N = \bigcup_{j,k,m\in\mathbb{N}} N_{j,k,m}.$$

Then N is a null set, and for every $x \in N^c$ the sequence $\{f_k(x) : k \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$. We define a measurable function $f : X \to \mathbb{R}$, unique up to pointwise a.e. equivalence, by

$$f(x) = \lim_{k\to\infty} f_k(x) \qquad \text{for } x \in N^c.$$

Letting $k \to \infty$ in (7.1), we find that for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that

$$|f_j(x) - f(x)| \le \frac{1}{m} \qquad \text{for } j \ge n \text{ and } x \in N^c.$$

It follows that f is essentially bounded and $f_j \to f$ in L^∞ as $j \to \infty$. This proves that L^∞ is complete. □

One useful consequence of this proof is worth stating explicitly.

Corollary 7.11. *Suppose that X is a measure space and $1 \le p < \infty$. If $\{f_k\}$ is a sequence in $L^p(X)$ that converges in L^p to f, then there is a subsequence $\{f_{k_j}\}$ that converges pointwise a.e. to f.*

As Example 4.26 shows, the full sequence need not converge pointwise a.e.

7.5. Duality

The dual space of a Banach space consists of all bounded linear functionals on the space.

Definition 7.12. If X is a real Banach space, the dual space of X^* consists of all bounded linear functionals $F : X \to \mathbb{R}$, with norm

$$\|F\|_{X^*} = \sup_{x\in X\setminus\{0\}} \left[\frac{|F(x)|}{\|x\|_X}\right] < \infty.$$

A linear functional is bounded if and only if it is continuous. For L^p spaces, we will use the Radon-Nikodym theorem to show that $L^p(X)^*$ may be identified with $L^{p'}(X)$ for $1 < p < \infty$. Under a σ-finiteness assumption, it is also true that $L^1(X)^* = L^\infty(X)$, but in general $L^\infty(X)^* \neq L^1(X)$.

Hölder's inequality implies that functions in $L^{p'}$ define bounded linear functionals on L^p with the same norm, as stated in the following proposition.

Proposition 7.13. *Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $1 < p \leq \infty$. If $f \in L^{p'}(X)$, then*

$$F(g) = \int fg\, d\mu$$

defines a bounded linear functional $F : L^p(X) \to \mathbb{R}$, and

$$\|F\|_{L^{p*}} = \|f\|_{L^{p'}}.$$

If X is σ-finite, then the same result holds for $p = 1$.

Proof. From Hölder's inequality, we have for $1 \leq p \leq \infty$ that

$$|F(g)| \leq \|f\|_{L^{p'}} \|g\|_{L^p},$$

which implies that F is a bounded linear functional on L^p with

$$\|F\|_{L^{p*}} \leq \|f\|_{L^{p'}}.$$

In proving the reverse inequality, we may assume that $f \neq 0$ (otherwise the result is trivial).

First, suppose that $1 < p < \infty$. Let

$$g = (\operatorname{sgn} f) \left(\frac{|f|}{\|f\|_{L^{p'}}} \right)^{p'/p}.$$

Then $g \in L^p$, since $f \in L^{p'}$, and $\|g\|_{L^p} = 1$. Also, since $p'/p = p' - 1$,

$$\begin{aligned} F(g) &= \int (\operatorname{sgn} f) f \left(\frac{|f|}{\|f\|_{L^{p'}}} \right)^{p'-1} d\mu \\ &= \|f\|_{L^{p'}}. \end{aligned}$$

Since $\|g\|_{L^p} = 1$, we have $\|F\|_{L^{p*}} \geq |F(g)|$, so that

$$\|F\|_{L^{p*}} \geq \|f\|_{L^{p'}}.$$

If $p = \infty$, we get the same conclusion by taking $g = \operatorname{sgn} f \in L^\infty$. Thus, in these cases the supremum defining $\|F\|_{L^{p*}}$ is actually attained for a suitable function g.

Second, suppose that $p = 1$ and X is σ-finite. For $\epsilon > 0$, let

$$A = \{x \in X : |f(x)| > \|f\|_{L^\infty} - \epsilon\}.$$

Then $0 < \mu(A) \leq \infty$. Moreover, since X is σ-finite, there is an increasing sequence of sets A_n of finite measure whose union is A such that $\mu(A_n) \to \mu(A)$, so we can find a subset $B \subset A$ such that $0 < \mu(B) < \infty$. Let

$$g = (\operatorname{sgn} f) \frac{\chi_B}{\mu(B)}.$$

Then $g \in L^1(X)$ with $\|g\|_{L^1} = 1$, and

$$F(g) = \frac{1}{\mu(B)} \int_B |f|\, d\mu \geq \|f\|_{L^\infty} - \epsilon.$$

It follows that

$$\|F\|_{L^{1*}} \geq \|f\|_{L^\infty} - \epsilon,$$

and therefore $\|F\|_{L^{1*}} \geq \|f\|_{L^\infty}$ since $\epsilon > 0$ is arbitrary. □

This proposition shows that the map $F = J(f)$ defined by

$$J : L^{p'}(X) \to L^p(X)^*, \qquad J(f) : g \mapsto \int fg\, d\mu, \tag{7.2}$$

is an isometry from $L^{p'}$ into L^{p*}. The main part of the following result is that J is onto when $1 < p < \infty$, meaning that every bounded linear functional on L^p arises in this way from an $L^{p'}$-function.

The proof is based on the idea that if $F : L^p(X) \to \mathbb{R}$ is a bounded linear functional on $L^p(X)$, then $\nu(E) = F(\chi_E)$ defines an absolutely continuous measure on $(X, \mathcal{A}, \mu)$, and its Radon-Nikodym derivative $f = d\nu/d\mu$ is the element of $L^{p'}$ corresponding to F.

Theorem 7.14 (Dual space of L^p)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. If $1 < p < \infty$, then (7.2) defines an isometric isomorphism of $L^{p'}(X)$ onto the dual space of $L^p(X)$.*

Proof. We just have to show that the map J defined in (7.2) is onto, meaning that every $F \in L^p(X)^*$ is given by $J(f)$ for some $f \in L^{p'}(X)$.

First, suppose that X has finite measure, and let

$$F : L^p(X) \to \mathbb{R}$$

be a bounded linear functional on $L^p(X)$. If $A \in \mathcal{A}$, then $\chi_A \in L^p(X)$, since X has finite measure, and we may define $\nu : \mathcal{A} \to \mathbb{R}$ by

$$\nu(A) = F(\chi_A).$$

If $A = \bigcup_{i=1}^\infty A_i$ is a disjoint union of measurable sets, then

$$\chi_A = \sum_{i=1}^\infty \chi_{A_i},$$

and the dominated convergence theorem implies that

$$\left\| \chi_A - \sum_{i=1}^n \chi_{A_i} \right\|_{L^p} \to 0$$

as $n \to \infty$. Hence, since F is a continuous linear functional on L^p,

$$\nu(A) = F(\chi_A) = F\left(\sum_{i=1}^\infty \chi_{A_i}\right) = \sum_{i=1}^\infty F(\chi_{A_i}) = \sum_{i=1}^\infty \nu(A_i),$$

meaning that ν is a signed measure on $(X, \mathcal{A})$.

If $\mu(A) = 0$, then χ_A is equivalent to 0 in L^p and therefore $\nu(A) = 0$ by the linearity of F. Thus, ν is absolutely continuous with respect to μ. By the Radon-Nikodym theorem, there is a function $f : X \to \mathbb{R}$ such that $d\nu = f d\mu$ and

$$F(\chi_A) = \int f\chi_A\, d\mu \qquad \text{for every} A \in \mathcal{A}.$$

Hence, by the linearity and boundedness of F,

$$F(\phi) = \int f\phi \, d\mu$$

for all simple functions ϕ, and

$$\left| \int f\phi \, d\mu \right| \leq M\|\phi\|_{L^p}$$

where $M = \|F\|_{L^{p*}}$.

Taking $\phi = \operatorname{sgn} f$, which is a simple function, we see that $f \in L^1(X)$. We may then extend the integral of f against bounded functions by continuity. Explicitly, if $g \in L^\infty(X)$, then from Theorem 7.8 there is a sequence of simple functions $\{\phi_n\}$ with $|\phi_n| \leq |g|$ such that $\phi_n \to g$ in L^∞, and therefore also in L^p. Since

$$|f\phi_n| \leq \|g\|_{L^\infty}|f| \in L^1(X),$$

the dominated convergence theorem and the continuity of F imply that

$$F(g) = \lim_{n\to\infty} F(\phi_n) = \lim_{n\to\infty} \int f\phi_n \, d\mu = \int fg \, d\mu,$$

and that

$$\left| \int fg \, d\mu \right| \leq M\|g\|_{L^p} \qquad \text{for every } g \in L^\infty(X). \tag{7.3}$$

Next we prove that $f \in L^{p'}(X)$. We will estimate the $L^{p'}$ norm of f by a similar argument to the one used in the proof of Proposition 7.13. However, we need to apply the argument to a suitable approximation of f, since we do not know *a priori* that $f \in L^{p'}$.

Let $\{\phi_n\}$ be a sequence of simple functions such that

$$\phi_n \to f \qquad \text{pointwise a.e. as } n \to \infty$$

and $|\phi_n| \leq |f|$. Define

$$g_n = (\operatorname{sgn} f) \left(\frac{|\phi_n|}{\|\phi_n\|_{L^{p'}}} \right)^{p'/p}.$$

Then $g_n \in L^\infty(X)$ and $\|g_n\|_{L^p} = 1$. Moreover, $fg_n = |fg_n|$ and

$$\int |\phi_n g_n| \, d\mu = \|\phi_n\|_{L^{p'}}.$$

It follows from these equalities, Fatou's lemma, the inequality $|\phi_n| \leq |f|$, and (7.3) that

$$\begin{aligned}
\|f\|_{L^{p'}} &\leq \liminf_{n\to\infty} \|\phi_n\|_{L^{p'}} \\
&\leq \liminf_{n\to\infty} \int |\phi_n g_n| \, d\mu \\
&\leq \liminf_{n\to\infty} \int |fg_n| \, d\mu \\
&\leq M.
\end{aligned}$$

Thus, $f \in L^{p'}$. Since the simple functions are dense in L^p and $g \mapsto \int fg\, d\mu$ is a continuous functional on L^p when $f \in L^{p'}$, it follows that $F(g) = \int fg\, d\mu$ for every $g \in L^p(X)$. Proposition 7.13 then implies that

$$\|F\|_{L^{p*}} = \|f\|_{L^{p'}},$$

which proves the result when X has finite measure.

The extension to non-finite measure spaces is straightforward, and we only outline the proof. If X is σ-finite, then there is an increasing sequence $\{A_n\}$ of sets with finite measure whose union is X. By the previous result, there is a unique function $f_n \in L^{p'}(A_n)$ such that

$$F(g) = \int_{A_n} f_n g\, d\mu \qquad \text{for all } g \in L^p(A_n).$$

If $m \geq n$, the functions f_m, f_n are equal pointwise a.e. on A_n, and the dominated convergence theorem implies that $f = \lim_{n\to\infty} f_n \in L^{p'}(X)$ is the required function.

Finally, if X is not σ-finite, then for each σ-finite subset $A \subset X$, let $f_A \in L^{p'}(A)$ be the function such that $F(g) = \int_A f_A g\, d\mu$ for every $g \in L^p(A)$. Define

$$M' = \sup\left\{\|f_A\|_{L^{p'}(A)} : A \subset X \text{ is } \sigma\text{-finite}\right\} \leq \|F\|_{L^p(X)^*},$$

and choose an increasing sequence of sets A_n such that

$$\|f_{A_n}\|_{L^{p'}(A_n)} \to M' \qquad \text{as } n \to \infty.$$

Defining $B = \bigcup_{n=1}^{\infty} A_n$, one may verify that f_B is the required function. □

A Banach space X is reflexive if its bi-dual X^{**} is equal to the original space X under the natural identification

$$\iota : X \to X^{**} \qquad \text{where } \iota(x)(F) = F(x) \text{ for every } F \in X^*,$$

meaning that x acting on F is equal to F acting on x. Reflexive Banach spaces are generally better-behaved than non-reflexive ones, and an immediate corollary of Theorem 7.14 is the following.

Corollary 7.15. *If X is a measure space and $1 < p < \infty$, then $L^p(X)$ is reflexive.*

Theorem 7.14 also holds if $p = 1$ provided that X is σ-finite, but we omit a detailed proof. On the other hand, the theorem does not hold if $p = \infty$. Thus L^1 and L^∞ are not reflexive Banach spaces, except in trivial cases.

The following example illustrates a bounded linear functional on an L^∞-space that does not arise from an element of L^1.

Example 7.16. Consider the sequence space $\ell^\infty(\mathbb{N})$. For

$$x = \{x_i : i \in \mathbb{N}\} \in \ell^\infty(\mathbb{N}), \qquad \|x\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |x_i| < \infty,$$

define $F_n \in \ell^\infty(\mathbb{N})^*$ by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

meaning that F_n maps a sequence to the mean of its first n terms. Then

$$\|F_n\|_{\ell^{\infty *}} = 1$$

for every $n \in \mathbb{N}$, so by the Alaoglu theorem on the weak-$*$ compactness of the unit ball, there exists a subsequence $\{F_{n_j} : j \in \mathbb{N}\}$ and an element $F \in \ell^\infty(\mathbb{N})^*$ with $\|F\|_{\ell^{\infty *}} \le 1$ such that $F_{n_j} \overset{*}{\rightharpoonup} F$ in the weak-$*$ topology on $\ell^{\infty *}$.

If $u \in \ell^\infty$ is the unit sequence with $u_i = 1$ for every $i \in \mathbb{N}$, then $F_n(u) = 1$ for every $n \in \mathbb{N}$, and hence

$$F(u) = \lim_{j\to\infty} F_{n_j}(u) = 1,$$

so $F \neq 0$; in fact, $\|F\|_{\ell^\infty} = 1$. Now suppose that there were $y = \{y_i\} \in \ell^1(\mathbb{N})$ such that

$$F(x) = \sum_{i=1}^\infty x_i y_i \qquad \text{for every } x \in \ell^\infty.$$

Then, denoting by $e_k \in \ell^\infty$ the sequence with kth component equal to 1 and all other components equal to 0, we have

$$y_k = F(e_k) = \lim_{j\to\infty} F_{n_j}(e_k) = \lim_{j\to\infty} \frac{1}{n_j} = 0$$

so $y = 0$, which is a contradiction. Thus, $\ell^\infty(\mathbb{N})^*$ is strictly larger than $\ell^1(\mathbb{N})$.

We remark that if a sequence $x = \{x_i\} \in \ell^\infty$ has a limit $L = \lim_{i\to\infty} x_i$, then $F(x) = L$, so F defines a generalized limit of arbitrary bounded sequences in terms of their Cesàro sums. Such bounded linear functionals on $\ell^\infty(\mathbb{N})$ are called Banach limits.

It is possible to characterize the dual of $L^\infty(X)$ as a space $\mathrm{ba}(X)$ of bounded, finitely additive, signed measures that are absolutely continuous with respect to the measure μ on X. This result is rarely useful, however, since finitely additive measures are not easy to work with. Thus, for example, instead of using the weak topology on $L^\infty(X)$, we can regard $L^\infty(X)$ as the dual space of $L^1(X)$ and use the corresponding weak-$*$ topology.