Problem Set 1: Math 206

Spring Quarter, 2011

1. Suppose that μ is a measure on a set X. If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence of measurable sets ($A_{i+1} \supset A_i$), show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

If $\{A_i : i \in \mathbb{N}\}$ is a decreasing sequence of measurable sets ($A_{i+1} \subset A_i$) and $\mu(A_1) < \infty$, show that

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

Give an example to show that the last result need not hold unless we assume that some set A_i has finite measure.

2. Suppose that $A \subset \mathbb{R}^n$ satisfies

$$\mu^*(R) = \mu^*(R \cap A) + \mu^*(R \setminus A)$$

for every rectangle R, where μ^* denotes Lebesgue outer measure. Prove that A is Lebesgue measurable.

3. (a) Show that the standard Cantor set has Lebesgue measure zero.

(b) Give an example of a subset of $[0, 1]$ that has non-zero Lebesgue measure but does not contain any nonempty open intervals.

(c) Show that the x-axis

$$E = \left\{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}$$

has zero (two-dimensional) Lebesgue measure.

4. Define the distance between sets $A, B \subset \mathbb{R}^n$ by

$$d(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}$$

where $|\cdot|$ denotes the Euclidean norm. If A, B are not necessarily measurable subsets of $\mathbb{R}^n$ and $d(A, B) > 0$, prove that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

where μ^* denotes Lebesgue outer measure.