Problem Set 2: Math 206
Spring Quarter, 2011

1. Prove that the unit open ball in $\mathbb{R}^2$,

$$B = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\},$$

cannot be expressed as a countable disjoint union of open rectangles.

2. Prove that every open subset of $\mathbb{R}^n$ is an F_σ and every closed subset is a G_δ.

3. Define the outer Jordan content $\overline{\kappa}(E)$ and inner Jordan content $\underline{\kappa}(E)$ of a set $E \subset \mathbb{R}^n$ by

$$\overline{\kappa}(E) = \inf\left\{\sum_{i=1}^{N} \mu(R_i) : \bigcup_{i=1}^{N} R_i \supset E,\ R_i \in \mathcal{R}(\mathbb{R}^n) \text{ almost disjoint}\right\},$$
$$\underline{\kappa}(E) = \sup\left\{\sum_{i=1}^{N} \mu(R_i) : E \supset \bigcup_{i=1}^{N} R_i,\ R_i \in \mathcal{R}(\mathbb{R}^n) \text{ almost disjoint}\right\},$$

where we use *finite* collections of almost disjoint rectangles.
(a) If $G \subset \mathbb{R}^n$ is open and $K \subset \mathbb{R}^n$ is compact, show that

$$\underline{\kappa}(G) = \mu(G), \qquad \overline{\kappa}(K) = \mu(K),$$

where μ denotes Lebesgue measure.
(b) A set $E \subset \mathbb{R}^n$ is Jordan measurable if $\overline{\kappa}(E) = \underline{\kappa}(E)$. Give an example of a subset of $[0, 1]$ that is Lebesgue measurable but not Jordan measurable.

4. Let A be a measurable subset of $\mathbb{R}$ with $\mu(A) > 0$. Show that for every $0 < \alpha < 1$ there is an open interval I such that

$$\mu(A \cap I) \geq \alpha\mu(I).$$

HINT. Choose an open set $U \supset A$ such that $\alpha\mu(U) \leq \mu(A)$, write U as a countable disjoint union of open intervals, and show that one of them has the desired property.