Problem Set 3: Math 206
Spring Quarter, 2011

1. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions on a measurable space $(X, \mathcal{A})$. If $A \in \mathcal{A}$ is a measurable set, define $h : X \to \overline{\mathbb{R}}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in A^c. \end{cases}$$

Show that h is measurable.

2. (a) Prove that the Borel σ-algebra $\mathcal{B}(\mathbb{C})$ on $\mathbb{C}$, with its standard metric topology, is generated by sets of the form

$$\{z = x + iy \in \mathbb{C} : -\infty < x < a,\ -\infty < y < b\}$$

where $a, b \in \mathbb{R}$.

(b) If $(X, \mathcal{A})$ is a measurable space, prove that a function $f = g + ih : X \to \mathbb{C}$ is measurable (in the sense that $f^{-1}(B) \in \mathcal{A}$ for every Borel subset $B \subset \mathbb{C}$) if and only if its real and imaginary parts $g, h : X \to \mathbb{R}$ are measurable.

3. True or false?

(a) If $|f| : \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue measurable, then $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue measurable.

(b) If $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ satisfy $f = g$ pointwise a.e. and f is Borel measurable, then g is Borel measurable.

(c) If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is sequentially lower semi-continuous, meaning that

$$f(x) \le \liminf_{n \to \infty} f(x_n)$$

for every sequence $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$, then f is Borel measurable.

4. (a) Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space (*i.e.* $\mu(X) < \infty$) and $f_n \to f$ pointwise where $f_n, f : X \to \mathbb{R}$ are real-valued functions. Show that for every $\epsilon > 0$ there exists a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

(b) Does this result remain true if $\mu(X) = \infty$ or if $f : X \to \overline{\mathbb{R}}$ is an extended real-valued function?