Problem Set 4: Math 206
Spring Quarter, 2011

1. If $f_1 \geq f_2 \geq f_3 \geq \dots$ is a monotone decreasing sequence of extended real-valued functions on a measure space and $\int f_1 \, d\mu < \infty$, show that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

Give an example to show this result need not be true if we omit the assumption that f_1 is integrable.

2. Prove the following generalization of the dominated convergence theorem: Let $\{f_n\}$ be a sequence of measurable, real-valued functions such that $f_n \to f$ pointwise as $n \to \infty$. If $|f_n| \leq g_n$, where g_n is integrable and

$$\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu$$

for some integrable function g, then

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

3. If f_n, f are integrable functions such that $f_n \to f$ uniformly on a finite measure space X, prove that $\int f_n \, d\mu \to \int f \, d\mu$. Use Egoroff's theorem (Problem 4 in Set 3) and the absolute continuity of the integral proved in class to give an alternative proof of the dominated convergence theorem.

4. Let $\{f_n\}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$

$$\int |f_n| \, d\mu \leq M \qquad \text{for all } n \in \mathbb{N}.$$

(a) Show that

$$\int |f| \, d\mu \leq M.$$

Give an example to show that we may have $\int |f_n| \, d\mu = M$ for every $n \in \mathbb{N}$ but $\int |f| \, d\mu < M$.
(b) Show that

$$\lim_{n\to\infty} \int \left| |f_n| - |f| - |f_n - f| \right| \, d\mu = 0.$$

Hint. Show that $||a + b| - |b|| \leq |a|$. (This quantifies the 'loss' of mass in the integral when $n \to \infty$.)

5. Let $(X, \mathcal{A}, \mu)$ be a complete measure space with finite measure, $\mu(X) < \infty$. Denote by $M(X)$ the space of (equivalence classes of) measurable functions $f : X \to \mathbb{R}$, where we identify functions that are equal μ-a.e.. For $f, g \in M(X)$, define

$$d(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} \, d\mu.$$

(a) Show that d is a metric on $M(X)$.

(b) A sequence $f_n \to f$ converges in measure if for every $\epsilon > 0$

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\right) = 0.$$

Show that $f_n \to f$ in measure if and only if $d(f_n, f) \to 0$ as $n \to \infty$.