Problem Set 5: Math 206
Spring Quarter, 2011

1. Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

with $f(0,0) = 0$. Compute the iterated integrals

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx, \qquad \int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy.$$

Why doesn't the result contradict Fubini's theorem?

2. Let $f : X \to [0, \infty]$ be a positive, measurable function on a σ-finite measure space $(X, \mathcal{A}, \mu)$. If $p > 0$, show that

$$\int f^p \, d\mu = p \int_0^\infty t^{p-1} \mu\left(\{x \in X : f(x) > t\}\right) dt.$$

3. Let $X = [0,1]$ with Lebesgue measure and $Y = [0,1]$ with counting measure. Give an example of an integrable function $f : X \times Y \to [0, \infty]$ for which Fubini's theorem does not apply. (This example shows that the theorem is not valid if the hypothesis of σ-finiteness is omitted.)

4. If $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable on $\mathbb{R}^n$, prove that the function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $F(x,y) = f(x - y)$ is Lebesgue measurable on $\mathbb{R}^n \times \mathbb{R}^n$.