Problem Set 6: Math 206

Spring Quarter, 2011

1. Suppose that $f_n, f \in L^p(X)$ and $f_n \to f$ pointwise a.e. Show that $\|f_n - f\|_{L^p} \to 0$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$. HINT. Use the generalization of the dominated convergence theorem stated in Problem 2, Set 4.

2. Suppose that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a measurable function on $\mathbb{R}^n$ equipped with Lebesgue measure μ. If f has maximal function $Mf : \mathbb{R}^n \to [0, \infty]$, prove that for any $0 < t < \infty$

$$\mu\left(\{x : Mf(x) > t\}\right) \leq \frac{2 \cdot 3^n}{t} \int_{|f(y)|>t/2} |f(y)|\, dy.$$

HINT. Apply the Hardy-Littlewood theorem to the function that is equal to f when $|f| > t/2$ and 0 otherwise.

3. If $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, where $1 \leq p < \infty$, prove that

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{|B_r(x)|} |f(y) - f(x)|^p \, dy = 0$$

pointwise a.e. HINT. Define

$$f^*(x) = \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{|B_r(x)|} |f(y) - f(x)|^p \, dy$$

and follow the proof of the Lebesgue differentiation theorem.

4. If ν is an arbitrary signed measure and μ is a σ-finite measure on a measurable space $(X, \mathcal{A})$ such that $\nu \ll \mu$, show that there exists a function $f : X \to \overline{\mathbb{R}}$ with well-defined, but possibly infinite, μ-integral such that

$$\nu(A) = \int_A f \, d\mu \qquad \text{for all } A \in \mathcal{A}.$$

(See Problem 3.14 in Folland for hints.) Give an example to show that this result need not be true if μ is not σ-finite.