CHAPTER 2

One Dimensional Dynamical Systems

We begin by analyzing some dynamical systems with one-dimensional phase spaces, and in particular their bifurcations. All equations in this Chapter are scalar equations. We mainly consider continuous dynamical systems on the real line $\mathbb{R}$, but we also consider continuous systems on the circle $\mathbb{T}$, as well as some discrete systems.

The restriction to one-dimensional systems is not as severe as it may sound. One-dimensional systems may provide a full model of some systems, but they also arise from higher-dimensional (even infinite-dimensional) systems in circumstances where only one degree of freedom determines their dynamics. Haken used the term 'slaving' to describe how the dynamics of one set of modes may follow the dynamics of some other, smaller, set of modes, in which case the behavior of the smaller set of modes determines the essential dynamics of the full system. For example, one-dimensional equations for the bifurcation of equilibria at a simple eigenvalue may be derived by means of the Lyapunov-Schmidt reduction.

2.1. Exponential growth and decay

> I said that population, when unchecked, increased in a geometrical ratio; and subsistence for man in an arithmetical ratio.
>
> — Thomas Malthus, 1798

The simplest ODE is the linear scalar equation

$$x_t = \mu x. \tag{2.1}$$

Its solution with initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{\mu t}$$

where the parameter μ is a constant. If $\mu > 0$, (2.1) describes exponential growth, and we refer to μ as the growth constant. The solution increases by a factor of e over time $T_e = 1/\mu$, and has doubling time

$$T = \frac{\log 2}{\mu}.$$

If $\mu = -\lambda < 0$, then (2.1) describes exponential decay, and we refer to $\lambda = |\mu|$ as the decay constant. The solution decreases by a factor of e over time $T_e = 1/\lambda$, and has half-life

$$T = \frac{\log 2}{\lambda}.$$

Note that μ or λ have the dimension of inverse time, as follows from equating the dimensions of the left and right hand sides of (2.1). The dimension of x is irrelevant since both sides are linear in x.

EXAMPLE 2.1. Malthus (1798) contrasted the potential exponential growth of the human population with the algebraic growth of resources needed to sustain it. An exponential growth law is too simplistic to accurately describe human populations. Nevertheless, after an initial lag period and before the limitation of nutrients, space, or other resources slows the growth, the population $x(t)$ of bacteria grown in a laboratory is well-described this law. The population doubles over the cell-division time. For example, *E. Coli* grown in glucose has a cell-division time of approximately 17 mins, corresponding to $\mu \approx 0.04\,\text{mins}^{-1}$.

EXAMPLE 2.2. Radioactive decay is well-described by (2.1) with $\mu < 0$, where $x(t)$ is the molar amount of radioactive isotope remaining after time t. For example, ^{14}C used in radioactive dating has a half-life of approximately 5730 years, corresponding to $\mu \approx -1.2 \times 10^{-4}\,\text{years}^{-1}$.

If $x_0 \geq 0$, then $x(t) \geq 0$ for all $t \in \mathbb{R}$, so this equation is consistent with modeling problems such as population growth or radioactive decay where the solution should remain non-negative. We assume also that the population or number of radioactive atoms is sufficiently large that we can describe it by a continuous variable.

The phase line of (2.1) consists of a globally asymptotically stable equilibrium $x = 0$ if $\mu < 0$, and an unstable equilibrium $x = 0$ if $\mu > 0$. If $\mu = 0$, then every point on the phase line is an equilibrium.

2.2. The logistic equation

The simplest model of population growth of a biological species that takes account of the effect of limited resources is the logistic equation

$$x_t = \mu x \left(1 - \frac{x}{K}\right). \tag{2.2}$$

Here, $x(t)$ is the population at time t, the constant $K > 0$ is called the carrying capacity of the system, and $\mu > 0$ is the maximum growth rate, which occurs at populations that are much smaller than the carrying capacity. For $0 < x < K$, the population increases, while for $x > K$, the population decreases.

We can remove the parameters μ, K by introducing dimensionless variables

$$\tilde{t} = \mu t, \qquad \tilde{x}(\tilde{t}) = \frac{x(t)}{K}.$$

Since $\mu > 0$, this transformation preserves the time-direction. The non-dimensionalized equation is $\tilde{x}_{\tilde{t}} = \tilde{x}\,(1 - \tilde{x})$ or, on dropping the tildes,

$$x_t = x(1 - x).$$

The solution of this ODE with initial condition $x(0) = x_0$ is

$$x(t) = \frac{x_0 e^t}{1 - x_0 + x_0 e^t}.$$

The phase line consists of an unstable equilibrium at $x = 0$ and a stable equilibrium at $x = 1$. For any initial data with $x_0 > 0$, the solution satisfies $x(t) \to 1$ as $t \to \infty$, meaning that the population approaches the carrying capacity.

2.3. The phase line

Consider a scalar ODE

$$x_t = f(x) \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

To sketch the phase line of this system one just has to examine the sign of f. Points where $f(x) = 0$ are equilibria. In intervals where $f(x) > 0$, solutions are increasing, and trajectories move to the right. In intervals where $f(x) < 0$, solutions are decreasing and trajectories move to the left. This gives a complete picture of the dynamics, consisting monotonically increasing or decreasing trajectories that approach equilibria, or go off to infinity.

The linearization of (2.3) at an equilibrium $\bar{x}$ is

$$x_t = ax, \qquad a = f'(\bar{x}).$$

The equilibrium is asymptotically unstable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$. If $f'(\bar{x}) = 0$, meaning that the fixed point is not hyperbolic, there is no immediate conclusion about the stability of $\bar{x}$, although one can determine its stability by looking at the behavior of the sign of f near the equilibrium.

Example 2.3. The ODE

$$x_t = x^2$$

with $f(x) = x^2$ has the unique equilibrium $x = 0$, but $f'(0) = 0$. Solutions with $x(0) < 0$ approach the equilibrium as $t \to \infty$, while solutions with $x(0) > 0$ leave it (and go off to ∞ in finite time). Such a equilibrium with one-sided stability is sometimes said to be semi-stable.

Example 2.4. For the ODE $x_t = -x^3$, the equilibrium $x = 0$ is asymptotically stable, while for $x_t = x^3$ it is unstable, even though $f'(0) = 0$ in both cases. Note, however, that perturbations from the equilibrium grow or decay algebraically in time, not exponentially as in the case of a hyperbolic equilibrium.

2.4. Bifurcation theory

> And thirdly, the code is more what you'd call "guidelines" than actual rules.
>
> Barbossa

Like the pirate code, the notion of a bifurcation is more of a guideline than an actual rule. In general, it refers to a qualitative change in the behavior of a dynamical system as some parameter on which the system depends varies continuously.

Consider a scalar ODE

$$x_t = f(x; \mu) \tag{2.4}$$

depending on a single parameter $\mu \in \mathbb{R}$ where f is a smooth function. The qualitative dynamical behavior of a one-dimensional continuous dynamical system is determined by its equilibria and their stability, so all bifurcations are associated with bifurcations of equilibria. One possible definition (which does not refer directly to the stability of the equilibria) is as follows.

Definition 2.5. A point (x_0, μ_0) is a bifurcation point of equilibria for (2.4) if the number of solutions of the equation $f(x; \mu) = 0$ for x in every neighborhood of (x_0, μ_0) is not a constant independent of μ.

The three most important one-dimensional equilibrium bifurcations are described locally by the following ODEs:

$$\begin{aligned} x_t &= \mu - x^2, && \text{saddle-node;} \\ x_t &= \mu x - x^2, && \text{transcritical;} \\ x_t &= \mu x - x^3, && \text{pitchfork.} \end{aligned} \tag{2.5}$$

We will study each of these in more detail below.

2.5. Saddle-node bifurcation

Consider the ODE

$$x_t = \mu + x^2. \tag{2.6}$$

Equations $x_t = \pm\mu \pm x^2$ with other choices of signs can be transformed into (2.6) by a suitable change in the signs of x and μ, although the transformation $\mu \mapsto -\mu$ changes increasing μ to decreasing μ.

The ODE (2.6)

$$x = \pm\sqrt{-\mu}$$

has two equilibria if $\mu < 0$, one equilibrium $x = 0$ if $\mu = 0$, and no equilibria if $\mu > 0$. For the function $f(x; \mu) = \mu + x^2$, we have

$$\frac{\partial f}{\partial x}(\pm\sqrt{-\mu}; \mu) = \pm 2\sqrt{-\mu}.$$

Thus, if $\mu < 0$, the equilibrium $\sqrt{-\mu}$ is unstable and the equilibrium $-\sqrt{-\mu}$ is stable. If $\mu = 0$, then the ODE is $x_t = x^2$, and $x = 0$ is a non-hyperbolic, semi-stable equilibrium.

This bifurcation is called a saddle-node bifurcation. In it, a pair of hyperbolic equilibria, one stable and one unstable, coalesce at the bifurcation point, annihilate each other and disappear.[1] We refer to this bifurcation as a subcritical saddle-node bifurcation, since the equilibria exist for values of μ below the bifurcation value 0. With the opposite sign $x_t = \mu - x^2$, the equilibria appear at the bifurcation point $(x, \mu) = (0, 0)$ as μ increases through zero, and we get a supercritical saddle-node bifurcation. Saddle-node bifurcations are the generic way that the number of equilibrium solutions of a dynamical system changes as some parameter is varied.

The name "saddle-node" comes from the corresponding two-dimensional bifurcation in the phase plane, in which a saddle point and a node coalesce and disappear, but the other dimension plays no essential role in that case and this bifurcation is one-dimensional in nature.

[1] If we were to allow complex equilibria, the equilibria would remain but become imaginary.

2.6. Transcritical bifurcation

Consider the ODE

$$x_t = \mu x - x^2.$$

This has two equilibria at $x = 0$ and $x = \mu$. For $f(x;\mu) = \mu x - x^2$, we have

$$\frac{\partial f}{\partial x}(x;\mu) = \mu - 2x, \qquad \frac{\partial f}{\partial x}(0;\mu) = \mu, \qquad \frac{\partial f}{\partial x}(\mu;\mu) = -\mu.$$

Thus, the equilibrium $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, while the equilibrium $x = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$. Note that although $x = 0$ is asymptotically stable for $\mu < 0$, it is not globally stable: it is unstable to negative perturbations of magnitude greater than μ, which can be small near the bifurcation point.

This transcritical bifurcation arises in systems where there is some basic "trivial" solution branch, corresponding here to $x = 0$, that exists for all values of the parameter μ. (This differs from the case of a saddle-node bifurcation, where the solution branches exist locally on only one side of the bifurcation point.). There is a second solution branch $x = \mu$ that crosses the first one at the bifurcation point $(x, \mu) = (0, 0)$. When the branches cross one solution goes from stable to unstable while the other goes from stable to unstable. This phenomenon is referred to as an "exchange of stability."

2.7. Pitchfork bifurcation

Consider the ODE

$$x_t = \mu x - x^3.$$

Note that this ODE is invariant under the reflectional symmetry $x \mapsto -x$. It often describes systems with this kind of symmetry *e.g.* systems where there is no distinction between left and right.

The system has one globally asymptotically stable equilibrium $x = 0$ if $\mu \leq 0$, and three equilibria $x = 0$, $x = \pm\sqrt{\mu}$ if μ is positive. The equilibria $\pm\sqrt{\mu}$ are stable and the equilibrium $x = 0$ is unstable for $\mu > 0$. Thus the stable equilibrium 0 loses stability at the bifurcation point, and two new stable equilibria appear. The resulting pitchfork-shape bifurcation diagram gives this bifurcation its name.

This pitchfork bifurcation, in which a stable solution branch bifurcates into two new stable branches as the parameter μ is increased, is called a supercritical bifurcation. Because the ODE is symmetric under $x \mapsto -x$, we cannot normalize all the signs in the ODE without changing the sign of t, which reverses the stability of equilibria.

Up to changes in the signs of x and μ, the other distinct possibility is the subcritical pitchfork bifurcation, described by

$$x_t = \mu x + x^3.$$

In this case, we have three equilibria $x = 0$ (stable), $x = \pm\sqrt{-\mu}$ (unstable) for $\mu < 0$, and one unstable equilibrium $x = 0$ for $\mu > 0$.

A supercritical pitchfork bifurcation leads to a "soft" loss of stability, in which the system can go to nearby stable equilibria $x = \pm\sqrt{\mu}$ when the equilibrium $x = 0$ loses stability as μ passes through zero. On the other hand, a subcritical pitchfork bifurcation leads to a "hard" lose of stability, in which there are no nearby equilibria and the system goes to some far-off dynamics (or perhaps to infinity) when the equilibrium $x = 0$ loses stability.

EXAMPLE 2.6. The ODE

$$x_t = \mu x + x^3 - x^5$$

has a subcritical pitchfork bifurcation at $(x, \mu) = (0, 0)$. When the solution $x = 0$ loses stability as μ passes through zero, the system can jump to one of the distant stable equilibria with

$$x^2 = \frac{1}{2}\left(1 + \sqrt{1 + 4\mu}\right),$$

corresponding to $x = \pm 1$ at $\mu = 0$.

2.8. The implicit function theorem

The above bifurcation equations for equilibria arise as normal forms from more general bifurcation equations, and they may be derived by a suitable Taylor expansion.

Consider equilibrium solutions of (2.4) that satisfy

$$f(x; \mu) = 0 \tag{2.7}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose that x_0 is an equilibrium solution at μ_0, meaning that

$$f(x_0; \mu_0) = 0.$$

Let us look for equilibria that are close to x_0 when μ is close to μ_0. Writing

$$x = x_0 + x_1 + \dots, \qquad \mu = \mu_0 + \mu_1$$

where x_1, μ_1 are small, and Taylor expanding (2.7) up to linear terms, we get that

$$\frac{\partial f}{\partial x}(x_0; \mu_0)x_1 + \frac{\partial f}{\partial \mu}(x_0; \mu_0)\mu_1 + \dots = 0$$

where the dots denote higher-order terms (*e.g.* quadratic terms). Hence, if

$$\frac{\partial f}{\partial x}(x_0; \mu_0) \neq 0$$

we expect to be able to solve (2.7) uniquely for x when (x, μ) is sufficiently close to (x_0, μ_0), with

$$x_1 = c\mu_1 + \dots, \qquad c = -\left[\frac{\partial f/\partial \mu(x_0; \mu_0)}{\partial f/\partial x(x_0; \mu_0)}\right]. \tag{2.8}$$

This is in fact true, as stated in the following fundamental result, which is the scalar version of the implicit function theorem.

THEOREM 2.7. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function and*

$$f(x_0; \mu_0) = 0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) \neq 0.$$

Then there exist $\delta, \epsilon > 0$ and a C^1 function

$$\bar{x} : (\mu_0 - \epsilon, \mu_0 + \epsilon) \to \mathbb{R}$$

such that $x = \bar{x}(\mu)$ is the unique solution of

$$f(x; \mu) = 0$$

with $|x - x_0| < \delta$ and $|\mu - \mu_0| < \epsilon$.

By differentiating this equation

$$f(\bar{x}(\mu);\mu) = 0$$

with respect to μ, setting $\mu = \mu_0$, and solving for $d\bar{x}/d\mu$, we get that

$$\frac{d\bar{x}}{d\mu}(\mu_0) = -\left.\frac{\partial f/\partial \mu}{\partial f/\partial x}\right|_{x=x_0,\mu=\mu_0}$$

in agreement with (2.8).

For the purposes of bifurcation theory, the most important conclusion from the implicit function theorem is the following:

Corollary 2.8. *If $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a C^1-function, then a necessary condition for a solution (x_0, μ_0) of (2.7) to be a bifurcation point of equilibria is that*

$$\frac{\partial f}{\partial x}(x_0;\mu_0) = 0. \tag{2.9}$$

Another way to state this result is that hyperbolic equilibria are stable under small variations of the system, and a local bifurcation of equilibria can occur only at a non-hyperbolic equilibrium.

While an equilibrium bifurcation is typical at points where (2.9) holds, there are exceptional, degenerate cases in which no bifurcation occurs. Thus, on its own, (2.9) is a necessary but not sufficient condition for the bifurcation of equilibria.

Example 2.9. The ODE

$$x_t = \mu - x^3$$

with $f(x;\mu) = \mu - x^3$ has a unique branch $x = (\mu)^{1/3}$ of globally stable equilibria. No bifurcation of equilibria occurs at $(0, 0)$ even though

$$\frac{\partial f}{\partial x}(0;0) = 0.$$

Note, however, that the equilibrium branch is not a C^1-function of μ at $\mu = 0$.

Example 2.10. The ODE

$$x_t = (\mu - x)^2$$

with $f(x;\mu) = (x-\mu)^2$ has a unique branch $x = \mu$ of non-hyperbolic equilibria, all of which are semi-stable. There are no equilibrium bifurcations, but

$$\frac{\partial f}{\partial x}(\mu;\mu) = 0$$

for all values of μ.

There is a close connection between the loss of stability of equilibria of (2.4) and their bifurcation. If $x = \bar{x}(\mu)$ is a branch of equilibria, then the equilibria are stable if

$$\frac{\partial f}{\partial x}(\bar{x}(\mu);\mu) < 0$$

and unstable if

$$\frac{\partial f}{\partial x}(\bar{x}(\mu);\mu) > 0.$$

It follows that if the equilibria lose stability at $\mu = \mu_0$, then $\partial f/\partial x(\bar{x}(\mu);\mu)$ changes sign at $\mu = \mu_0$ so (2.9) holds at that point. Thus, the loss of stability of a branch of equilibria due to the passage of an eigenvalue of the linearized system through zero is typically associated with the appearance or disappearance of other equilibria.[2]

When (2.9) holds, we have to look at the higher-order terms in the Taylor expansion of $f(x;\mu)$ to determine what type of bifurcation (if any) actually occurs. We can always transfer a bifurcation point at (x_0,μ_0) to $(0,0)$ by the change of variables $x\mapsto x-x_0$, $\mu\mapsto\mu-\mu_0$. Moreover, if $x=\bar{x}(\mu)$ is a solution branch, then $x\mapsto x-\bar{x}(\mu)$ maps the branch to $x=0$.

Let us illustrate the idea with the simplest example of a saddle-node bifurcation. Suppose that

$$f(0,0)=0, \qquad \frac{\partial f}{\partial x}(0,0)=0$$

so that $(0,0)$ is a possible bifurcation point. Further suppose that

$$\frac{\partial f}{\partial \mu}(0;0)=a\neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(0;0)=b\neq 0.$$

Then Taylor expanding $f(x;\mu)$ up to the leading-order nonzero terms in x, μ we get that

$$f(x;\mu)=a\mu+\frac{1}{2}bx^2+\dots$$

Thus, neglecting the higher-order terms,[3] we may approximate the ODE (2.4) near the origin by

$$x_t=a\mu+\frac{1}{2}bx^2.$$

By rescaling x and μ, we may put this ODE in the standard normal form for a saddle-node bifurcation. The signs of a and b determine whether the bifurcation is subcritical or supercritical and which branches are stable or unstable. For example, if $a,b>0$, we get the same bifurcation diagram and local dynamics as for (2.6).

As in the case of the implicit function theorem, this formal argument does not provide a rigorous proof that a saddle-node bifurcation occurs, and one has to justify the neglect of the higher-order terms. In particular, it may not be obvious which terms can be safely neglected and which terms must be retained. We will not give any further details here, but simply summarize the resulting conclusions in the following theorem.

Theorem 2.11. *Suppose that $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a smooth function and (x_0,μ_0) satisfy the necessary bifurcation conditions*

$$f(x_0,\mu_0)=0, \qquad \frac{\partial f}{\partial x}(x_0;\mu_0)=0.$$

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0)\neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0;\mu_0)\neq 0$$

then a saddle-node bifurcation occurs at (x_0,μ_0)

[2]In higher-dimensional systems, an equilibrium may lose stability by the passage of a complex conjugate pair of eigenvalues across the real axis. This does not lead to an equilibrium bifurcation, since the eigenvalues are always nonzero. Instead, as we will discuss later on, it is typically associated with the appearance or disappearance of periodic solutions in a Hopf bifurcation.

[3]For example, μx is small compared with μ and x^3 is small compared with x^2 since x is small, and μ^2 is small compared with μ since μ is small.

- *If*
$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0;\mu_0) \neq 0$$
then a transcritical bifurcation occurs at (x_0, μ_0).
- *If*
$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x^2}(x_0;\mu_0) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^3 f}{\partial x^3}(x_0;\mu_0) \neq 0$$
then a pitchfork bifurcation occurs at (x_0, μ_0)

The conditions in the theorem are rather natural; they state that the leading nonzero terms in the Taylor expansion of f agree with the terms in the corresponding normal form. Note that a saddle-node bifurcation is generic, in the sense that other derivatives of f have to vanish at the bifurcation point if a saddle-node bifurcation is not to occur.

In each case of Theorem 2.11, one can find local coordinates near (x_0, μ_0) that put the equation $f(x; \mu) = 0$ in the normal form for the corresponding bifurcation in a sufficiently small neighborhood of the bifurcation point. In particular, the bifurcation diagrams look locally like the ones considered above. The signs of the nonzero terms determine the stability of the various branches and whether or not the bifurcation is subcritical or supercritical.

Example 2.12. Bifurcation points for the ODE
$$x_t = \mu x - e^x$$
must satisfy
$$\mu x - e^x = 0, \qquad \mu - e^x = 0,$$
which implies that $(x, \mu) = (1, e)$. This can also be seen by plotting the graphs of $y = \mu x$ and $y = e^x$: the line $y = \mu x$ is tangent to the curve $y = e^x$ at $(x, y) = (1, e)$ when $\mu = e$. Writing
$$x = 1 + x_1 + \dots, \qquad \mu = e + \mu_1 + \dots$$
we find that the Taylor approximation of the ODE near the bifurcation point is
$$x_{1t} = \mu_1 - \frac{e}{2} x_1^2 + \dots.$$
Thus, there a supercritical saddle node bifurcation at $(x, \mu) = (1, e)$. For $\mu > e$, the equilibrium solutions are given by
$$x = 1 \pm \sqrt{\frac{2}{e}(\mu - e)} + \dots.$$
The solution with $x > 1$ is stable, while the solution with $x < 1$ is unstable.

Example 2.13. The ODE
$$x_t = \mu^2 + \mu x - x^3 \tag{2.10}$$
with $f(x; \mu) = \mu^2 + \mu x - x^3$ has a supercritical pitchfork bifurcation at $(x, \mu) = (0, 0)$, since it satisfies the conditions of the theorem. The quadratic term μ^2 does not affect the type of bifurcation. We will return to this equation in Example 2.14 below.

2.9. Buckling of a rod

Consider two rigid rods of length L connected by a torsional spring with spring constant k and subject to a compressive force of strength λ. If x is the angle of the rods to the horizontal, then the potential energy of the system is

$$V(x) = \frac{1}{2}kx^2 + 2\lambda L\,(\cos x - 1)\,.$$

Here $kx^2/2$ is the energy required to compress the spring by an angle $2x$ and $2\lambda L\,(1 - \cos x)$ is the work done on the system by the external force. Equilibrium solutions satisfy $V'(x) = 0$ or

$$x - \mu \sin x = 0, \qquad \mu = \frac{2\lambda L}{k}$$

where μ is a dimensionless force parameter.

The equation has the trivial, unbuckled, solution branch $x = 0$. Writing

$$f(x;\mu) = -V'(x) = \mu \sin x - x,$$

the necessary condition for an equilibrium bifurcation to occur on this branch is

$$\frac{\partial f}{\partial x}(0,\mu) = \mu - 1 = 0$$

which occurs at $\mu = 1$. The Taylor expansion of $f(x;\mu)$ about $(0,1)$ is

$$f(x;\mu) = (\mu - 1)x - \frac{1}{6}x^3 + \dots$$

Thus, there is a supercritical pitchfork bifurcation at $(x,\mu) = (0,1)$. The bifurcating equilbria near this point are given for $0 < \mu - 1 \ll 1$ by

$$x = \sqrt{6(\mu - 1)} + \dots.$$

This behavior can also be seen by sketching the graphs of $y = x$ and $y = \mu \sin x$. Note that the potential energy V goes from a single well to a double well as μ passes through 1.

This one-dimensional equation provides a simple model for the buckling of an elastic beam, one of the first bifurcation problems which was originally studied by Euler (1757).

2.10. Imperfect bifurcations

According to Theorem 2.11, a saddle-node bifurcation is the generic bifurcation of equilibria for a one-dimensional system, and additional conditions are required at a bifurcation point to obtain a transcritical or pitchfork bifurcation. As a result, these latter bifurcations are not structurally stable and can be destroyed by arbitrarily small perturbations that break the conditions under which they occur

First, let us consider a perturbed, or imperfect, pitchfork bifurcation that is described by

$$x_t = \lambda + \mu x - x^3 \tag{2.11}$$

where $(\lambda,\mu) \in \mathbb{R}^2$ are real parameters. Note that if $\lambda = 0$, this system has the reflectional symmetry $x \mapsto -x$ and a pirchfork bifurcation, but this symmetry is broken when $\lambda \neq 0$.

The cubic polynomial $p(x) = \lambda + \mu x - x^3$ has repeated roots if

$$\lambda + \mu x - x^3 = 0, \qquad \mu - 3x^2 = 0$$

which occurs if $\mu = 3x^2$ and $\lambda = -2x^3$ or

$$4\mu^3 = 27\lambda^2.$$

As can be seen by sketching the graph of p, there are three real roots if $\mu > 0$ and

$$27\lambda^2 < 4\mu^3,$$

and one real root if $27\lambda^2 > 4\mu^3$. The surface of the roots as a function of (λ, μ) forms a cusp catastrophe.

If $\lambda \neq 0$, the pitchfork bifurcation is perturbed to a stable branch that exists for all values of μ without any bifurcations and a supercritical saddle-node bifurcation in which the remaining stable and unstable branches appear.

EXAMPLE 2.14. The ODE (2.10) corresponds to (2.11) with $\lambda = \mu^2$. As this parabola passes through the origin in the (λ, μ)-plane, we get a supercritical pitchfork bifurcation at $(x, \mu) = (0, 0)$. We then get a further saddle-node bifurcation at $(x, \mu) = (-2/9, 4/27)$ when the parabola $\lambda = \mu^2$ crosses the curve $4\mu^3 = 27\lambda^2$.

Second, consider an imperfect transcritical bifurcation described by

$$x_t = \lambda + \mu x - x^2 \tag{2.12}$$

where $(\lambda, \mu) \in \mathbb{R}^2$ are real parameters. Note that if $\lambda = 0$, this system has the equilibrium solution $x = 0$, but if $\lambda < 0$ there is no solution branch that is defined for all values of μ.

The equilibrium solutions of (2.12) are

$$x = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 + 4\lambda}\right),$$

which are real provided that $\mu^2 + 4\lambda \geq 0$. If $\lambda < 0$, the transcritical bifurcation for $\lambda = 0$ is perturbed into two saddle-node bifurcations at $\mu = \pm 2\sqrt{-\lambda}$; while if $\lambda > 0$, we get two non-intersecting solution branches, one stable and one unstable, and no bifurcations occur as μ is varied.

2.11. Dynamical systems on the circle

Problems in which the dependent variable $x(t) \in \mathbb{T}$ is an angle, such as the phase of an oscillation, lead to dynamical systems on the circle.

As an example, consider a forced, highly damped pendulum. The equation of motion of a linearly damped pendulum of mass m and length ℓ with angle $x(t)$ to the vertical acted on by a constant angular force F is

$$m\ell x_{tt} + \delta x_t + mg \sin x = F$$

where δ is a positive damping coefficient and g is the acceleration due to gravity.

The damping coefficient δ has the dimension of Force $\times$ Time. For motions in which the damping and gravitational forces are important, an appropriate time scale is therefore δ/mg, and we introduce a dimensionless time variable

$$\tilde{t} = \frac{mg}{\delta} t, \qquad \frac{d}{dt} = \frac{mg}{\delta}\frac{d}{d\tilde{t}}.$$

The angle x is already dimensionless, so we get the non-dimensionalized equation

$$\epsilon x_{\tilde{t}\tilde{t}} + x_{\tilde{t}} + \sin x = \mu$$

where the dimensionless parameters ϵ, μ are given by

$$\epsilon = \frac{m^2 g\ell}{\delta^2}, \qquad \mu = \frac{F}{mg}.$$

For highly damped motions, we neglect the term $\epsilon x_{\tilde{t}\tilde{t}}$ and set $\epsilon = 0$. Note that one has to be careful with such an approximation: the higher-order derivative is a singular perturbation, and it may have a significant effect even though its coefficient is small. For example, the second order ODE with $\epsilon > 0$ requires two initial conditions, whereas the first order ODE for $\epsilon = 0$ requires only one. Thus for the reduced equation with $\epsilon = 0$ we can specify the initial location of the pendulum, but we cannot specify its initial velocity, which is determined by the ODE. We will return to such questions in more detail later on, but for now we simply set $\epsilon = 0$.

Dropping the tilde on $\tilde{t}$ we then get the ODE

$$x_t = \mu - \sin x,$$

where μ is a nondimensionalized force parameter. If $\mu = 0$, this system has two equilibria: a stable one at $x = 0$ corresponding to the pendulum hanging down, and an unstable one at $\mu = \pi$ corresponding to the pendulum balanced exactly above its fulcrum. As μ increases, these equilibria move toward each other (the stable equilibrium is 'lifted up' by the external force), and when $\mu = 1$ they coalesce and disappear in a saddle-node bifurcation at $(x, \mu) = (\pi/2, 1)$. For $\mu > 1$, there are no equilibria. The external force is sufficiently strong to overcome the gravitational force and the pendulum rotates continuously about its fulcrum.

2.12. Discrete dynamical systems

A one-dimensional discrete dynamical system

$$x_{n+1} = f(x_n) \tag{2.13}$$

is given by iterating a map f, which we assume is smooth. Its equilibria are fixed points $\bar{x}$ of f such that

$$\bar{x} = f(\bar{x}).$$

The orbits, or trajectories, of (2.13) consist of a sequence of points x_n rather than a curve $x(t)$ as in the case of an ODE. As a result, there are no topological restrictions on trajectories of a discrete dynamical systems, and unlike the continuous case there is no simple, general way to determine their phase portrait. In fact, their behavior may be extremely complex, as the logistic map discussed in Section 2.15 illustrates.

There is a useful graphical way to sketch trajectories of (2.13): Draw the graphs $y = f(x)$, $y = x$ and iterate points vertically to $y = f(x)$, which updates the state, and horizontally to $y = x$, which updates x-value.

The simplest discrete dynamical system is the linear scalar equation

$$x_{n+1} = \mu x_n. \tag{2.14}$$

The solution is

$$x_n = \mu^n x_0.$$

If $\mu \neq 1$, the origin $x = 0$ is the unique fixed point of the system. If $|\mu| < 1$, this fixed point is globally asymptotically stable, while if $|\mu| > 1$ it is unstable. Note that if $\mu > 0$, successive iterates approach or leave the origin monotonically, while if $\mu < 0$, they alternate on either side of the origin. If $\mu = 1$, then every point is a fixed point of (2.14), while if $\mu = -1$, then every point has period two. The map is invertible if $\mu \neq 0$ when orbits are defined backward and forward in time. If $\mu = 0$, every point is mapped to the origin after one iteration, and orbits are not defined backward in time.

EXAMPLE 2.15. The exponential growth of a population of bacteria that doubles every generation is described by (2.14) with $\mu = 2$ where x_n denotes the population of the nth generation.

The linearization of (2.13) about a fixed point $\bar{x}$ is

$$x_{n+1} = a x_n, \qquad a = f'(\bar{x})$$

where the prime denotes an x-derivative. We say that the fixed point is hyperbolic if $|f'(\bar{x})| \neq 1$, and in that case it is stable if

$$|f'(\bar{x})| < 1$$

and unstable if

$$|f'(\bar{x})| > 1.$$

As for continuous dynamical systems, the stability of non-hyperbolic equilibria (with $f'(\bar{x}) = \pm 1$) cannot be determined solely from their linearization.

After fixed points, the next simplest type of solution of (2.13) consists of periodic solutions. A state x_1 or x_2 has period two if the system has an orbit of the form $\{x_1, x_2\}$ where

$$x_2 = f(x_1), \qquad x_1 = f(x_2).$$

The system oscillates back and forth between the two states x_1, x_2.

We can express periodic orbits as fixed points of a suitable map. We write the composition of f with itself as $f^2 = f \circ f$, meaning that

$$f^2(x) = f\left(f(x)\right).$$

Note that this is not the same as the square of f — for example $\sin^2(x) = \sin(\sin x)$ not $(\sin x)^2$ — but our use of the notation should be clear from the context. If $\{x_1, x_2\}$ is a period-two orbit of f, then x_1, x_2 are fixed points of f^2 since

$$f^2(x_1) = f\left(f(x_1)\right) = f(x_2) = x_1.$$

Conversely, if x_1 is a fixed point of f^2 and $x_2 = f(x_1)$, then $\{x_1, x_2\}$ is a period-two orbit of (2.13).

More generally, for any $N \in \mathbb{N}$, a period-N orbit of (2.13) consists of points $\{x_1, x_2, x_3, \dots, x_N\}$ such that

$$x_2 = f(x_1), \qquad x_3 = f(x_2), \qquad \dots, \qquad x_1 = f(x_N).$$

In that case, each x_i is an N-periodic solution of (2.13) and is a fixed point of f^N, the N-fold composition of f with itself. If a point has period N, then it also has period equal to every positive integer multiple of N. For example, a fixed point has period equal to every positive integer, while a point with period two has period equal to every even positive integer. If x_1 is a periodic solution of (2.13), the minimal period of x_1 is the smallest positive integer N such that $f^N(x_1) = x_1$. Thus, for example, the fixed points of f^4 include all fixed points of f and all two-periodic points, as well as all points whose minimal period is four.

2.13. Bifurcations of fixed points

Next, we consider some bifurcations of a one-dimensional discrete dynamical system

$$x_{n+1} = f(x_n; \mu) \tag{2.15}$$

depending on a parameter $\mu \in \mathbb{R}$. As usual, we assume that f is a smooth function. In addition to local bifurcations of fixed points that are entirely analogous to bifurcations of equilibria in continuous dynamical systems, these systems possess a period-doubling, or flip, bifurcation that has no continuous analog. They also possess other, more complex, bifurcations. First, we consider bifurcations of fixed points.

If x_0 is a fixed point of (2.15) at $\mu = \mu_0$, then by the implicit function theorem the fixed-point equation

$$f(x; \mu) - x = 0$$

is uniquely solvable for x close to x_0 and μ close to μ_0 provided that

$$\frac{\partial f}{\partial x}(x_0; \mu_0) - 1 \neq 0.$$

Thus, necessary conditions for (x_0, μ_0) to be a bifurcation point of fixed points are that

$$f(x_0; \mu_0) = x_0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) = 1.$$

The following typical bifurcations at $(x, \mu) = (0, 0)$ are entirely analogous to the ones in (2.5), in which we replace the equation $f(x; \mu) = 0$ for equilibria by the equation $f(x; \mu) - x = 0$ for fixed points:

$$\begin{aligned}
x_{n+1} &= \mu + x_n - x_n^2, && \text{saddle-node;}\\
x_{n+1} &= (1 + \mu)x_n - x_n^2, && \text{transcritical;}\\
x_{n+1} &= (1 + \mu)x_n - x_n^3, && \text{pitchfork.}
\end{aligned}$$

For completeness, we state the corresponding theorem for fixed-point bifurcations.

Theorem 2.16. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function and (x_0, μ_0) satisfies the necessary condition for the bifurcation of fixed points for (2.15):*

$$f(x_0, \mu_0) = x_0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) = 1.$$

- *If*
$$\frac{\partial f}{\partial \mu}(x_0; \mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0; \mu_0) \neq 0$$
then a saddle-node bifurcation occurs at (x_0, μ_0)
- *If*
$$\frac{\partial f}{\partial \mu}(x_0; \mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x \partial \mu}(x_0; \mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0; \mu_0) \neq 0$$
then a transcritical bifurcation occurs at (x_0, μ_0).

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x^2}(x_0;\mu_0) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^3 f}{\partial x^3}(x_0;\mu_0) \neq 0$$

then a pitchfork bifurcation occurs at (x_0, μ_0)

2.14. The period-doubling bifurcation

Suppose that (2.15) has a branch of fixed points $x = \bar{x}(\mu)$ such that

$$\bar{x}(\mu) = f(\bar{x}(\mu);\mu).$$

The fixed point can lose stability in two ways: (a) the eigenvalue $f_x(\bar{x}(\mu);\mu)$ passes through 1; (b) the eigenvalue $f_x(\bar{x}(\mu);\mu)$ passes through -1. In the first case, we typically get a bifurcation of fixed points, but in the second case the implicit function theorem implies that no such bifurcation occurs. Instead, the loss of stability is typically associated with the appearance or disappearance of period two orbits near the fixed point.

To illustrate this, we consider the following system

$$x_{n+1} = -(1+\mu)x_n + x_n^3$$

with

$$f(x;\mu) = -(1+\mu)x + x^3$$

The fixed points satisfy $x = -(1+\mu)x + x^3$, whose solutions are $x = 0$ and

$$x = \pm\sqrt{2+\mu} \tag{2.16}$$

for $\mu > -2$. We have

$$\frac{\partial f}{\partial x}(0;\mu) = -(1+\mu),$$

so the fixed point $x = 0$ is stable if $-2 < \mu < 0$ and unstable if $\mu > 0$ or $\mu < -2$.

The eigenvalue $f_x(0;\mu)$ passes through 1 at $\mu = -2$, and $x = 0$ gains stability at a supercritical pitchfork bifurcation in which the two new fixed points (2.16) appear. Note that for $\mu > -2$

$$\frac{\partial f}{\partial x}(\pm\sqrt{2+\mu};\mu) = 5 + 2\mu > 1$$

so these new fixed points are unstable.

The eigenvalue $f_x(0;\mu)$ passes through -1 at $\mu = 0$, and $x = 0$ loses stability at that point. As follows from the implicit function theorem, there is no bifurcation of fixed points: the only other branches of fixed points are (2.16), equal to $x = \pm\sqrt{2}$ at $\mu = 0$, which are far away from $x = 0$. Instead, we claim that a new orbit of period two appears at the bifurcation point.

To show this, we analyze the fixed points of the two-fold composition of f

$$\begin{aligned} f^2(x;\mu) &= -(1+\mu)\left[-(1+\mu)x + x^3\right] + \left[-(1+\mu)x + x^3\right]^3 \\ &= (1+\mu)^2 x - (1+\mu)(2+2\mu+\mu^2)x^3 + 3(1+\mu)^2 x^5 - 3(1+\mu)x^7 + x^9. \end{aligned}$$

The period-doubling bifurcation for f corresponds to a pitchfork bifurcation for f^2. Near the bifurcation point $(x,\mu) = (0,0)$, we may approximate f^2 by

$$f^2(x;\mu) = (1+2\mu)x - 2x^3 + \dots.$$

The fixed points of f^2 are therefore given approximately by

$$x = \pm\sqrt{\mu},$$

and they are stable. The corresponding stable period-two orbit is $\{\sqrt{\mu}, -\sqrt{\mu}\}$.

If the original equation was

$$x_{n+1} = -(1+\mu)x_n + ax_n^2 + bx_n^3,$$

with a quadratically nonlinear term instead of a cubically nonlinear term, we still get a pitchfork bifurcation in f^2 at $(x, \mu) = (0, 0)$. We find similarly that if

$$f(x; \mu) = -(1+\mu)x + ax^2 + bx^3,$$

then

$$f^2(x; \mu) = (1+2\mu)x - 2(a^2+b)x^3 + \dots.$$

Thus, f^2 has a pitchfork bifurcation provided that $a^2 + b \neq 0$. Noting that

$$a = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0; 0), \qquad b = \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(0; 0)$$

this result leads to the following theorem.

THEOREM 2.17. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function and (x_0, μ_0) satisfies*

$$f(x_0, \mu_0) = x_0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) = -1.$$

If

$$\frac{\partial^2 f}{\partial x \partial \mu}(x_0; \mu_0) \neq 0, \qquad \frac{1}{2}\left[\frac{\partial^2 f}{\partial^2 x}(x_0; \mu_0)\right]^2 + \frac{1}{3}\frac{\partial^3 f}{\partial^3 x}(x_0; \mu_0) \neq 0$$

then a period-doubling bifurcation for (2.15) occurs at (x_0, μ_0).

2.15. The logistic map

The discrete logistic equation is

$$x_{n+1} = \mu x_n (1 - x_n), \tag{2.17}$$

which is (2.13) with

$$f(x; \mu) = \mu x(1-x).$$

We can interpret (2.17) as a model of population growth in which x_n is the population of generation n. In general, positive values of x_n may map to negative values of x_{n+1}, which would not make sense when using the logistic map as a population model. We will restrict attention to $1 \le \mu \le 4$, in which case $f(\cdot; \mu)$ maps points in $[0, 1]$ into $[0, 1]$. Then the population x_{n+1} is $\mu(1 - x_n)$ times the population x_n of the previous generation. If $0 \le x_n < 1 - 1/\mu$, the population increases, whereas if If $1 - 1/\mu < x_n \le 1$, the population decreases. Superficially, (2.17) may appear similar to the logistic ODE (2.2), but its qualitative properties are very different. In particular, note that the quadratic logistic map is not monotone or invertible on $[0, 1]$ and is typically two-to-one.

Equation (2.17) has two branches of fixed points,

$$x = 0, \qquad x = 1 - \frac{1}{\mu}.$$

We have

$$\frac{\partial f}{\partial x}(x, \mu) = \mu(1 - 2x)$$

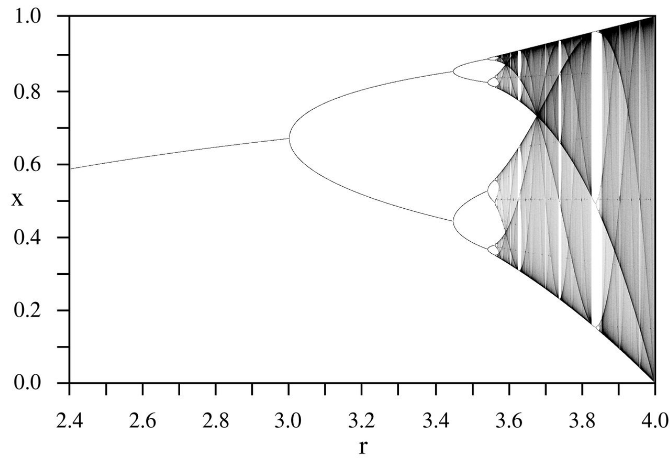

FIGURE 1. A bifurcation diagram for the logistic map where $r = \mu$. (From Wikipedia.)

so that

$$\frac{\partial f}{\partial x}(0, \mu) = \mu, \qquad \frac{\partial f}{\partial x}\left(1 - \frac{1}{\mu}, \mu\right) = 2 - \mu.$$

Thus, the fixed point $x = 0$ is unstable for $\mu > 1$. The fixed point $x = 1 - 1/\mu$ is stable for $1 < \mu < 3$ and unstable $\mu > 3$. There is a transcritical bifurcation of fixed points at $(x, \mu) = (0, 1)$ where these two branches exchange stability. The fixed point $x = 1 - 1/\mu$ loses stability at $\mu = 3$ in a supercritical period doubling bifurcation as $f_x = 2 - \mu$ passes through -1.

We will not carry out a further analysis of (2.17) here. We note, however, that there is a sequence of supercritical period doubling bifurcations corresponding to the appearance of stable periodic orbits of order $2, 4, 8, \dots, 2^k, \dots$ at $\mu = \mu_k$. These bifurcation points have a finite limit

$$\mu_\infty = \lim_{k \to \infty} \mu_k \approx 3.570.$$

The first few values are approximately given by

$$\mu_1 = 3, \qquad \mu_2 = 3.449, \qquad \mu_3 = 3.544, \qquad \mu_4 = 3.564, \dots.$$

The bifurcation values approach their limit μ_∞ geometrically, with

$$\lim_{k \to \infty} \left(\frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}\right) = 4.6692\dots$$

where $4.6692\dots$ is a universal Feigenbaum constant. For $\mu_\infty < \mu \leq 4$, the logistic map is chaotic, punctated by windows in which it has an asymptotically stable periodic orbit of (rather remarkably) period three.

A bifurcation diagram for (2.17) is shown in Figure 1.

2.16. References

We have studied some basic examples of equilibrium bifurcations, but have not attempted to give a general analysis, which is part of singularity or catastrophe theory. Further introductory discussions can be found in [**4, 11**]. For a systematic account, including equilibrium bifurcations in the presence of symmetry, see [**5**].