CHAPTER 3

Two Dimensional Linear Systems of ODEs

A (first-order, autonomous, homogeneous) linear system of two ODEs has the form

$$x_t = ax + by, \qquad y_t = cx + dy$$

where a, b, c, d are (real) constants. The matrix form is

$$\vec{x}_t = A\vec{x} \tag{3.1}$$

where

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.2}$$

The initial condition for (3.1) is

$$\vec{x}(0) = \vec{x}_0 \tag{3.3}$$

where

$$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

and $x_0, y_0 \in \mathbb{R}$ are arbitrary constants.

3.1. The exponential of a linear map

The solution of the linear, autonomous, scalar IVP

$$x_t = ax, \qquad x(0) = x_0$$

may be written as $x(t) = e^{ta}x_0$. An analogous formula holds for systems with a suitable definition of the exponential of a linear map, or matrix. The definition does not depend on the dimension of the system, so we initially consider the d-dimensional case.

Definition 3.1. If $A : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map, or its corresponding matrix, and $t \in \mathbb{R}$, then the exponential $e^{tA} : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{n!}t^nA^n + \ldots. \tag{3.4}$$

The series (3.4) is a natural generalization of the Taylor series of the scalar exponential function. It makes sense if A is a $d \times d$ matrix, so that its powers are well-defined, and then the series defines a $d \times d$ matrix e^{tA}.

We remark that it is useful conceptually to distinguish between a linear map, which is a geometrical object like a rotation of vectors, and the matrix that represents it with respect to some basis. The matrix of a given linear map depends on the basis, not only on the map. We consider linear maps acting on $\mathbb{R}^d$ with its standard basis, so usually we will not distinguish between linear maps and matrices, but one should still be able to view the results we discuss from a geometrical perspective.

We summarize some properties of the matrix exponential in the following result.

THEOREM 3.2. *For every linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, or its corresponding matrix, the series (3.4) converges absolutely for all $t \in \mathbb{R}$. It is a differentiable function of t, and*

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A. \tag{3.5}$$

Moreover, $e^{0A} = I$ and

$$e^{sA}e^{tA} = e^{(s+t)A}$$

for all $s, t \in \mathbb{R}$, and e^{tA} is invertible with

$$\left(e^{tA}\right)^{-1} = e^{-tA}.$$

The proof of these results is similar to the proof of the analogous results for the usual scalar exponential function. For example, term by term differentiation of the series (which one can show is justified) gives

$$\begin{aligned} \frac{d}{dt}e^{tA} &= A + tA^2 + \cdots + \frac{1}{(n-1)!}t^{n-1}A^n + \dots \\ &= A\left(I + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{(n-1)!}t^{n-1}A^{n-1} + \dots\right) \\ &= Ae^{tA}. \end{aligned}$$

Similarly, if A, B commute (meaning that $AB = BA$) then multiplication and rearrangement of the series for e^A, e^B implies that

$$e^Ae^B = e^{A+B}.$$

Note, however, that this relation does *not* hold for non-commuting maps or matrices; this is a significant difference from the scalar case.

It follows from (3.5) that the solution of the IVP (3.1), (3.3) is given by

$$\vec{x}(t) = e^{tA}\vec{x}_0. \tag{3.6}$$

To verify this, note that if $\vec{x}(t)$ is given by (3.6) then

$$\frac{d}{dt}\vec{x}(t) = Ae^{tA}\vec{x}_0 = A\vec{x}(t),$$

and $\vec{x}(0) = \vec{x}_0$ since $e^{0A} = I$. Thus, (3.6) is the unique solution of (3.1), and

$$\Phi_t = e^{tA}$$

is the flow map associated with the linear vector field $A\vec{x}$. Note that the flow map is also linear, as follows from the superposition property of linear equations.

Let us consider some simple 2×2 examples.

EXAMPLE 3.3. The system

$$x_t = -x, \qquad y_t = -y$$

with

$$A = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has a vector field pointing radially inward toward the origin. The solutions are

$$x(t) = x_0 e^{-t}, \qquad y(t) = y_0 e^{-t},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

In this case, $A = -I$ and $A^n = (-1)^n I$, so that

$$\begin{aligned} e^{tA} &= I - tI + \frac{1}{2!}t^2 I^2 + \cdots + \frac{(-1)^n}{n!}t^n I + \dots \\ &= \left(1 - t + \frac{1}{2!}t^2 + \cdots + \frac{(-1)^n}{n!}t^n + \dots\right) I \\ &= e^{-t} I \end{aligned}$$

or

$$\exp\left[-t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

in agreement with the above solution. The trajectories consist of radial lines that approach 0 as $t \to \infty$, and the equilibrium point 0. The flow map is a contraction by a factor e^{-t}.

Example 3.4. The solution of

$$x_t = -x, \qquad y_t = y$$

with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is

$$x(t) = x_0 e^{-t}, \qquad y(t) = y_0 e^{t}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

In this case,

$$A^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} e^{tA} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots + \frac{t^n}{n!} \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix} + \dots \\ &= \begin{pmatrix} 1 - t + \cdots + \frac{(-1)^n t^n}{n!} + \dots & 0 \\ 0 & 1 + t + \cdots + \frac{t^n}{n!} + \dots \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}. \end{aligned}$$

Thus,

$$\exp\left[t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix},$$

in agreement with the above solution. The trajectories consists of branches of the hyperbolas $xy =$ constant, the positive and negative x and y axes, and the equilibrium point 0. The x-values approach 0 as $t \to \infty$ while the y-values go off to infinity as $t \to \infty$ (unless $y_0 = 0$). The flow map is a contraction by e^{-t} in the x-direction and an expansion by e^t in the y-direction.

EXAMPLE 3.5. The system

$$x_t = -y, \qquad y_t = x$$

with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has a vector field that is orthogonal to the radial vector field. Eliminating y, we get $x_{tt} + x = 0$, and the solutions are

$$x(t) = x_0 \cos t - y_0 \sin t, \qquad y(t) = x_0 \sin t + y_0 \cos t,$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

In this case,

$$A^{2n} = (-1)^n I, \qquad A^{2n+1} = (-1)^n A,$$

and using the Taylor series for $\cos t$, $\sin t$ we get

$$\begin{aligned} e^{tA} &= I + tA - \frac{1}{2!}t^2 I - \frac{1}{3!}t^3 A + \frac{1}{4!}t^4 I + \dots \\ &= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \dots\right) I + \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots\right) A \\ &= (\cos t) I + (\sin t) A. \end{aligned}$$

Thus,

$$\exp\left[t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

in agreement with the solution above. The trajectories consist of circles centered at the origin, and the equilibrium 0. All solutions are 2π-periodic functions of t, and the flow map is a counter-clockwise rotation about the origin through an angle t.

EXAMPLE 3.6. The system

$$x_t = y, \qquad y_t = 0$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has the solution

$$x(t) = x_0 + y_0 t, \qquad y(t) = y_0,$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

In this case, $A^n = 0$ for $n \geq 2$ (a matrix whose powers are eventually zero is said to be nilpotent) and (3.4) shows, in agreement with this solution, that

$$\exp\left[t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The x-axis consists of equilibrium, and the other trajectories are lines $y = y_0$ with $y_0 \neq 0$. The flow map consists of a shear by t in the x-direction.

It is not easy to use the series definition to compute the exponential of a matrix directly except in special cases. We can however use a similarity transformation to reduce a general matrix to a canonical form whose matrix exponential can be easily computed.

PROPOSITION 3.7. *Suppose that $A = P^{-1}BP$ is similar to a matrix B by a nonsingular matrix P. Then the matrix exponential of A*

$$e^{tA} = P^{-1}e^{tB}P$$

is similar to the matrix exponential of B

PROOF. We have

$$A^n = P^{-1}B^nP$$

because of the cancelation in the inner factors $P^{-1}P = I$. Thus,

$$\begin{aligned} e^{tA} &= I + tA + \dots + \frac{1}{n!}t^nA^n + \dots \\ &= I + tP^{-1}BP + \dots + \frac{1}{n!}t^nP^{-1}B^nP + \dots \\ &= P^{-1}\left(I + tB + \dots + \frac{1}{n!}t^nB^n + \dots\right)P \\ &= P^{-1}e^{tB}P. \end{aligned}$$

□

A matrix is said to be diagonalizable if it is similar to a diagonal matrix Λ. In the 2×2 case, this means that

$$A = P^{-1}\Lambda P, \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1, λ_2 are the (not necessarily distinct) eigenvalues of A. We compute from the series definition that

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

and

$$e^{t\Lambda} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

Thus, the exponential of a diagonalizable 2×2 matrix is given by

$$e^{tA} = P^{-1}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}P.$$

Most matrices are diagonalizable. Sufficient conditions for the diagonalizability of a matrix are: (a) it has simple eigenvalues; or (b) it is symmetric, in which case its eigenvalues are real and there is an orthonormal basis of eigenvectors. There exist, however, nondiagonalizable matrices.

In the case of 2×2 matrices, it follows from the Jordan canonical form that every nondiagonalizable matrix A is similar to a 2×2 Jordan block

$$A = P^{-1}JP, \qquad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where λ is the (necessarily repeated) eigenvalue of A. To compute the exponential of J, note that

$$J = \Lambda + N, \qquad \Lambda = \lambda I, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Since Λ is a multiple of the identity, it commutes with N and therefore

$$e^{tJ} = e^{t\Lambda} e^{tN}.$$

The series definition implies that $e^{t\Lambda} = e^{\lambda t} I$ and, from Example 3.6,

$$e^{tN} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Thus, the exponential of a nondiagonalizable 2×2 matrix is given by

$$e^{tA} = e^{\lambda t} P^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P.$$

3.2. Eigenvalues and eigenvectors

We can represent the solution of a general linear system (3.1) in terms of the eigenvalues and (generalized) eigenvectors of A. The result is equivalent to the representation of the matrix exponential in terms of a similarity transformation.[1]

For simplicity, we consider the 2×2 case where A is given by (3.2), although the ideas generalize in a fairly straightforward way to systems of any dimension.

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a nonzero eigenvector $\vec{r} \in \mathbb{C}^2$ such that

$$A\vec{r} = \lambda \vec{r}.$$

The eigenvalues are solutions of the characteristic equation of A,

$$\det(A - \lambda I) = 0.$$

In that case, a solution of (3.1) is given by

$$\vec{x}(t) = e^{\lambda t} \vec{r}.$$

If the 2×2 matrix A is diagonalizable, then it has has (possibly repeated) eigenvalues $\{\lambda_1, \lambda_2\}$ with linearly independent eigenvectors $\{\vec{r}_1, \vec{r}_2\}$. The general solution of (3.1) is given by

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{r}_1 + c_2 e^{\lambda_2 t} \vec{r}_2.$$

To satisfy the initial condition (3.3), we choose the scalar constants $c_1, c_2 \in \mathbb{C}$ so that

$$\vec{x}_0 = c_1 \vec{r}_1 + c_2 \vec{r}_2.$$

There exist unique such constants because $\{\vec{r}_1, \vec{r}_2\}$ are linearly independent and hence form a basis of $\mathbb{C}^2$.

It is convenient here to allow real or complex eigenvalues even though the system is real. Any complex eigenvalues come in complex-conjugate pairs, and we can take their eigenvectors to be complex conjugates also. In that case the constants c_1, c_2 are complex, but for real-valued initial data c_1, c_2 are complex conjugates and the solution $\vec{x}(t)$ is also real-valued.

[1]The columns of the similarity matrix P^{-1} consist of (generalized) eigenvectors of A.

If A is nondiagonalizable with a repeated eigenvalue λ, then it has linearly independent generalized eigenvectors $\{\vec{r}, \vec{s}\}$ such that

$$(A - \lambda I)\vec{r} = 0, \qquad (A - \lambda I)\vec{r} = \vec{s}.$$

the solution of the ODE is

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{r} + c_2 e^{\lambda t}\left(t\vec{r} + \vec{s}\right)$$

where the scalar constants c_1, c_2 are chosen so that

$$\vec{x}_0 = c_1\vec{r} + c_2\vec{s}.$$

3.3. Classification of 2×2 linear systems

The linear system (3.1) has the equilibrium $\vec{x} = 0$, and this is the only equilibrium if A is nonsingular. We classify the equilibrium of the 2×2 linear system (3.1) in terms of the (possibly repeated) eigenvalues λ_1, λ_2 of A as follows.

Saddle point: If λ_1, λ_2 are real and of opposite signs.
Node: If λ_1, λ_2 are real and of the same sign. Stable if $\lambda_1, \lambda_2 < 0$ and unstable if $\lambda_1, \lambda_2 > 0$.
Spiral point: If $\lambda_1, \lambda_2 = \tau \pm i\omega$ are a complex conjugate pair with nonzero real part. Stable if $\tau < 0$ and unstable if $\tau > 0$.
Center: If $\lambda_1, \lambda_2 = \pm i\omega$ are a pure imaginary, complex conjugate pair.
Singular: If λ_1 or λ_2 is zero, when there is a one or two dimensional subspace of equilibria.

For instance, Example 3.3 is a stable node, Example 3.4 is a saddle point, Example 3.5 is a center, and Example 3.6 is singular. A stable node or stable spiral is asymptotically stable. A center is stable but not asymptotically stable. A saddle point, unstable node or unstable spiral is unstable.

The eigenvalues $\lambda = \lambda_1, \lambda_2$ of the 2×2 matrix A in (3.2) satisfy

$$\lambda^2 - \tau\lambda + D = 0$$

where τ is the trace of A and D is the determinant of A,

$$\tau = \operatorname{tr} A = a + d = \lambda_1 + \lambda_2, \qquad D = \det A = ad - bc = \lambda_1\lambda_2.$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4D}\right).$$

We therefore get the following criteria for the type of the equilibrium 0 in terms of the trace and determinant of A.

Saddle point: If $D < 0$.
Node: If $0 < 4D \leq \tau^2$. Stable if $\tau < 0$ and unstable if $\tau > 0$.
Spiral point: If $\tau \neq 0$ and $4D > \tau^2$. Stable if $\tau < 0$ and unstable if $\tau > 0$.
Center: If $\tau = 0$ and $D > 0$.
Singular: If $D = 0$.

We distinguish between hyperbolic and non-hyperbolic equilibria according to the following definition.

DEFINITION 3.8. The equilibrium $\vec{x} = 0$ of (3.1) is hyperbolic if no eigenvalue of A has zero real part.

Thus, a saddle point, node or spiral point is hyperbolic, but a center or singular point is not. Hyperbolic equilibria are robust under small perturbations (such as the inclusion of non-linear terms), but non-hyperbolic equilibria are not.

The stable (respectively, unstable) subspace E^s (respectively, E^u) of 0 is the subspace spanned by the (generalized) eigenvectors associated with eigenvectors whose eigenvalues have negative (respectively, positive) real parts. Thus, $\vec{x}_0 \in E^s$ if the solution approaches the equilibrium forward in time,

$$e^{tA}\vec{x}_0 \to 0 \qquad \text{as } t \to \infty,$$

while $\vec{x}_0 \in E^u$ if the solution approaches the equilibrium backward in time,

$$e^{tA}\vec{x}_0 \to 0 \qquad \text{as } t \to -\infty.$$

(Note that this is not equivalent to the condition that $e^{tA}\vec{x}_0 \to \infty$ as $t \to \infty$.) The center subspace E^c of 0 is the subspace spanned by (generalized) eigenvectors with zero real part. The corresponding solutions may remain bounded or grow algebraically in time.

For example, a stable node or spiral point of a 2×2 linear system has $E^s = \mathbb{R}^2$ and $E^u = E^c = \{0\}$; a center has $E^c = \mathbb{R}^2$ and $E^u = E^s = \{0\}$; and a saddle point has one-dimensional stable and unstable subspaces E^s, E^u and $E^c = \{0\}$. The center subspace of any hyperbolic equilibrium is $\{0\}$.

3.4. The linear oscillator

Consider a mass $m > 0$ on a Hookean spring with spring constant $k > 0$ subject to linear damping with coefficient $\delta \geq 0$. The displacement $x(t)$ of the mass from equilibrium satisfies

$$mx_{tt} + \delta x_t + kx = 0. \tag{3.7}$$

Introducing the momentum $y = mx_t$, we may write this as first order system

$$x_t = \frac{1}{m}y, \qquad y_t = -kx - \delta y$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 1/m \\ -k & -\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of this system are

$$\lambda = \frac{1}{2}\left[-\delta \pm \sqrt{\delta^2 - \frac{4k}{m}}\right].$$

Depending on the strength of the damping, we have three possibilities for the equilibrium:

: Undamped When $\delta = 0$, $(x, y) = 0$ is a center. All solutions are periodic with angular frequency $\omega = \sqrt{k/m}$.

: Underdamped When $0 < \delta < 2\sqrt{km}$, $(x, y) = 0$ is a stable spiral point. All solutions decay to zero in an oscillatory fashion as $t \to \infty$.

: Overdamped When $\delta > 2\sqrt{km}$, $(x, y) = 0$ is a stable node. All solutions eventually decay monotonically to zero (with the displacement x passing through zero at most once).

The damping $\delta = 2\sqrt{km}$ at which the oscillator switches from underdamped to overdamped is called critical damping. Note that this equation is dimensionally consistent: both δ and $\sqrt{km}$ have the dimension of Mass/Time.

Exactly the same analysis applies to linear electrical circuits. Suppose a circuit consists of an inductor with inductance L, a resistor with resistance R, and a capacitor with capacitance C, placed in series. If $I(t)$ is the current in the circuit and $Q(t)$ is the charge on the capacitor, then the change in voltage across the inductor is

$$\Delta V_L = LI_t,$$

the change in voltage across the resistor is given by Ohm's law

$$\Delta V_R = RI,$$

and the change of voltage across the capacitor is

$$\Delta V_C = \frac{1}{C}Q.$$

According to Kirchoff's law the total change in voltage around the circuit is zero, meaning that

$$\Delta V_L + \Delta V_L + \Delta V_L = 0.$$

Moreover, since charge is conserved, we have $I = Q_t$. It follows that $Q(t)$ satisfies

$$LQ_{tt} + RQ_t + \frac{1}{C}Q = 0.$$

This equation is identical to (3.7) with L playing the role of mass, R the role of the damping constant, and $1/C$ the role of the spring constant.

3.5. Flow maps and areas

An important aspect of flows is how they affect areas (or volumes) in phase space. For example, do areas contract or increase under the flow or do they remain the same? We consider this briefly here in the special case of 2×2 linear systems.

The oriented area $|\mathcal{P}|$ of the parallelogram $\mathcal{P}$ spanned by vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ is given by

$$|\mathcal{P}| = \det\left[\vec{x}, \vec{y}\right].$$

If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, then the oriented area of the parallelogram $A(\mathcal{P})$ spanned by the vectors $A\vec{x}, A\vec{y} \in \mathbb{R}^2$ is given by

$$\begin{aligned} |A(\mathcal{P})| &= \det\left[A\vec{x}, A\vec{y}\right] \\ &= \det\left(A\left[\vec{x}, \vec{y}\right]\right) \\ &= (\det A)(\det\left[\vec{x}, \vec{y}\right]) \\ &= (\det A)|\mathcal{P}| \end{aligned}$$

Thus, $\det A$ is the factor by which A multiplies oriented areas.

The following result related the determinant of a matrix exponential to the trace of the matrix. Exactly the same result is true for a linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ in any number of dimensions d, as can be shown using the general Jordan canonical form, but we state and prove the theorem only for the case $d = 2$.

Theorem 3.9. *If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, then*

$$\det e^{tA} = e^{t \operatorname{tr} A}.$$

PROOF. If A is diagonalizable, then

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P,$$

and

$$e^{tA} = P^{-1} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P.$$

Hence

$$\det e^{tA} = \det P^{-1} \cdot \det \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \cdot \det P = e^{(\lambda_1+\lambda_2)t} = e^{t \operatorname{tr} A}.$$

If A is non-diagonalizable, then

$$A = P^{-1} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} P,$$

and

$$\det e^{tA} = \det P^{-1} \cdot \det \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \cdot \det P = e^{2\lambda t} = e^{t \operatorname{tr} A}.$$

□

3.6. References

For a classification of equilibria based on real canonical forms, see [**9**].