Sample Final Questions
Math 207A, Fall 2011

1. Sketch the phase plane of the ODE

$$x_{tt} + x\left(x^2 - 1\right)^2 = 0.$$

Find the equilibria and determine their stability. Are any of the equilibria hyperbolic?

2. Find the equilibria of the system

$$\begin{aligned} x_t &= 2y, \\ y_t &= 2x - 3x^2 - y\left(x^3 - x^2 + y^2 - \mu\right). \end{aligned}$$

Linearize the equations about each equilibrium and classify them. What local bifurcations occur as the parameter μ varies?

3. (a) Write the system

$$\begin{aligned} x_t &= x - y - x\left(x^2 + y^2\right), \\ y_t &= x + y - y\left(x^2 + y^2\right) \end{aligned}$$

in polar coordinates and sketch the phase plane. How do solutions behave at $t \to \infty$?

(b) Define a Poincaré return map $P : (0, \infty) \to (0, \infty)$ as follows: for $x > 0$, $(P(x), 0)$ is the next intersection point of the trajectory starting at $(x, 0)$ with the positive x-axis. By solving the polar equations, show that

$$P(x) = \frac{cx}{\sqrt{1 + (c^2 - 1)x^2}}$$

where $c = e^{2\pi}$.

(c) Find the fixed point $\bar{x} \in (0, \infty)$ of P and determine its stability.