Sample Final Questions
Math 207A, Fall 2011
Brief Solutions

1. Sketch the phase plane of the ODE

$$x_{tt} + x\left(x^2 - 1\right)^2 = 0.$$

Find the equilibria and determine their stability. Are any of the equilibria hyperbolic?

Solution

- This is a conservative system $x_{tt} + V'(x) = 0$ with potential

$$V(x) = \frac{1}{6}(x^2 - 1)^3.$$

- There are three equilibria $x = 0, \pm 1$. None of them are hyperbolic. The equilibrium $x = 0$ is a nondegenerate minimum of $V(x)$ and is a center. The equilibria $x = \pm 1$ are degenerate critical points of $V(x)$. All of the orbits are periodic except for two heteroclinic orbits, one in the upper half of the (x, x_t)-plane that goes from $(-1, 0)$ to $(1, 0)$, the other in the lower half plane that goes from $(1, 0)$ to $(-1, 0)$

2. Find the equilibria of the system

$$
\begin{aligned}
x_t &= 2y, \\
y_t &= 2x - 3x^2 - y\left(x^3 - x^2 + y^2 - \mu\right).
\end{aligned}
$$

Linearize the equations about each equilibrium and classify them. What local bifurcations occur as the parameter μ varies?

Solution

- The equilibria are
$$(x, y) = (0, 0), \quad (2/3, 0).$$

- The equilibrium $(0, 0)$ is a saddle for all values of μ (with eigenvalues $\lambda = \pm 1$).

- The linearization at $(2/3, 0)$ is
$$x_t = 2y, \qquad y_t = -2x + \mu' y$$
where
$$\mu' = \mu + \frac{4}{27}.$$
The eigenvalues are
$$\lambda = \frac{1}{2}\left(\mu' \pm \sqrt{(\mu')^2 - 16}\right).$$
This is a stable node if $\mu' \leq -4$, , a stable spiral if $-4 < \mu' < 0$, a (linearized) center if $\mu' = 0$, an unstable spiral if $0 < \mu' < 4$, and an unstable node if $4 \leq \mu'$.

- No local bifurcation occurs at $\mu' = \pm 4$; the equilibrium remains hyperbolic and a node simply turns into a spiral (the local flows are topologically conjugate). There is a Hopf bifurcation at $\mu' = 0$ when a pair of complex-conjugate eigenvalues crosses the imaginary axis. By the Hopf bifurcation theorem, there is a one-parameter family of periodic orbits near $(x, y, \mu') = (2/3, 0, 0)$.

- Note that there are no local bifurcations at $\mu = 0$, or $\mu' = 4/27$, but there is a global homoclinic bifurcation at $\mu = 0$.

3. (a) Write the system

$$x_t = x - y - x\left(x^2 + y^2\right),$$
$$y_t = x + y - y\left(x^2 + y^2\right)$$

in polar coordinates and sketch the phase plane. How do solutions behave at $t \to \infty$?

(b) Define a Poincaré return map $P : (0, \infty) \to (0, \infty)$ as follows: for $x > 0$, $(P(x), 0)$ is the next intersection point of the trajectory starting at $(x, 0)$ with the positive x-axis. By solving the polar equations, show that

$$P(x) = \frac{cx}{\sqrt{1 + (c^2 - 1)x^2}}$$

where $c = e^{2\pi}$.

(c) Find the fixed point $\bar{x} \in (0, \infty)$ of P and determine its stability.

Solution

- Part (a) was in a previous problem set. The result is

$$r_t = r - r^3, \qquad \theta_t = 1.$$

- (c) Since $\theta = t + \theta_0$, the next intersection with the x-axis occurs after time $t = 2\pi$. The solution of the ODE for $r(t)$ with initial condition $r(0) = x$ is

$$r(t) = \frac{e^t x}{\sqrt{1 + (e^{2t} - 1)x^2}}.$$

The Poincaré map is $P(x) = r(2\pi)$ which gives the result.

- The fixed point of $P(x)$ is $x = 1$ and $0 < P'(1) < 1$, which means that it is a stable fixed point. This fixed point of P corresponds to the stable limit cycle $r = 1$ of the original ODE.