Problem set 1
Math 207A, Fall 2011
Solutions

1. Write the IVP for the forced, damped pendulum

$$\begin{aligned} &x_{tt} + \delta x_t + \omega_0^2 \sin x = \gamma \cos \omega t, \\ &x(0) = x_0, \qquad x_t(0) = v_0 \end{aligned}$$

as an IVP for an autonomous first-order system. What is the dimension of the system?

Solution

- Introduce variables (x, v, s) where $v = x_t$, $s = t$. Then

$$\begin{aligned} x_t &= v, \\ v_t &= -\delta v - \omega_0^2 \sin x + \gamma \cos \omega s, \\ s_t &= 1, \end{aligned}$$

  with initial conditions

$$x(0) = x_0, \qquad v(0) = v_0, \qquad s(0) = 0.$$

Remark. Since the phase space is three-dimensional, chaotic behavior is possible, and indeed it occurs in suitable parameter regimes.

2. Solve the scalar IVP

$$x_t = x(\log x)^\alpha, \qquad x(0) = x_0$$

where $\alpha > 0$ and $x_0 > 1$. Find the maximal time-interval on which the solution exists. For what values of α does the solution exist for all times?

Solution

- Separating variables, we get

  $$\int \frac{1}{x(\log x)^\alpha}\, dx = \int dt$$

  To compute the x-integral, use the substitution $u = \log x$, which gives

  $$\int \frac{1}{u^\alpha}\, du = t + C$$

  where C is a constant of integration.

- If $\alpha \neq 1$, the solution is

  $$\frac{1}{1-\alpha} u^{1-\alpha} = t + C.$$

  The initial condition implies that

  $$C = \frac{1}{1-\alpha}(\log x_0)^{1-\alpha},$$

  so

  $$x(t) = \exp\left\{\left[(1-\alpha)t + (\log x_0)^{1-\alpha}\right]^{1/(1-\alpha)}\right\}.$$

  Note that $\log x_0 > 0$ since $x_0 > 1$.

- If $0 < \alpha < 1$, the solution exists for all $0 \leq t < \infty$.

- If $\alpha > 1$ the solution exists only for a finite time interval $0 \leq t < T$ where $T > 0$ is given by

  $$T = \frac{1}{(\alpha - 1)(\log x_0)^{\alpha-1}}.$$

- If $\alpha = 1$, the solution is

$$x(t) = \exp\left[(\log x_0) \exp t\right].$$

  The solution exist for all t, although it grows very rapidly (doubly exponentially) as $t \to \infty$.

Remark. Note that the solution of $x_t = x \log x$ exists globally in time even though the right-hand side grows faster than a linear function of x, albeit by a slowly growing logarithmic factor. Any higher power of $\log x$, however, leads to solutions that blow up in finite time.

3. The position $x(t) \in \mathbb{R}$ of a particle of mass m moving in one space dimension in a potential $V(x)$ satisfies

$$mx_{tt} = -V'(x)$$

where the prime denotes a derivative with respect to x. Show that the total energy

$$\frac{1}{2}mx_t^2 + V(x) = \text{constant}$$

is conserved. What can you say about the time-interval of existence of solutions for: (a) the attractive potential $V(x) = x^4$; (b) the repulsive potential $V(x) = -x^4$?

Solution

- Using the chain rule and the ODE, we get

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}mx_t^2 + V(x)\right] &= mx_t x_{tt} + V'(x)x_t \\ &= -x_t V'(x) + V'(x)x_t \\ &= 0. \end{aligned}$$

  Hence the total energy is constant.

- If $V(x) = x^4$ then

$$\frac{1}{2}mx_t^2 + x^4 = \text{constant},$$

  which implies that both x, x_t are bounded functions of time. The extension theorem, applied to the corresponding first-order systems for (x, x_t), then implies that the solutions exist globally for all $t \in \mathbb{R}$.

- In fact, if $V(x) = x^4$, all non-zero solutions are periodic functions of t (as is, strictly speaking, the zero solution).

- If $V(x) = -x^4$ then

$$\frac{1}{2}mx_t^2 - x^4 = \text{constant},$$

  but this does not imply that x, x_t remain bounded, so there is no conclusion from the extension theorem.

- In fact, if $V(x) = -x^4$ and $\frac{1}{2}mx_t^2 - x^4 = E_0$, where

$$E_0 = \frac{1}{2}mv_0^2 - x_0^4, \qquad x(0) = x_0, \quad x_t(0) = v_0,$$

  then $x(t)$ satisfies the first order ODE

$$x_t = \pm\sqrt{\frac{2}{m}\,(E_0 + x^4)}, \qquad x(0) = x_0$$

  with an appropriate choice of the sign.

- If $E_0 \neq 0$, then solutions of this ODE (whose right hand side grows like x^2) go off to infinity in finite time both as $t \to -\infty$ and $t \to \infty$. The (unstable) equilibrium solution $x(t) = 0$ exists for all time. Finally, if $E_0 = 0$ and $x(t) \neq 0$, then: when x_0, v_0 have the same sign, solutions go off to infinity in finite time as $t \to \infty$ and approach 0 as $t \to -\infty$; when x_0, v_0 have the opposite sign, solutions approach 0 as $t \to \infty$ and blow up at finite negative time.

Remark. The previous statements may be easier to follow if you sketch the (x, x_t)-phase plane of the system (as we'll do in class later on).

4. Linearize the Lorenz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z \end{aligned}$$

about the equilibrium solution $(x, y, z) = (0, 0, 0)$. Show that this equilibrium is linearly stable if $r < 1$ and linearly unstable if $r > 1$.

Solution

- We obtain the linearized system is by neglecting the quadratically nonlinear terms, which gives

  $$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y, \\ z_t &= -\beta z. \end{aligned}$$

  In matrix form, this system is

  $$\vec{x}_t = A\vec{x}$$

  where $\vec{x} = (x, y, z)^T$ and

  $$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

- The equilibrium $(0, 0, 0)^T$ is stable if all eigenvalues of A have negative real part and unstable if some eigenvalue of A has positive real part.

- We have

  $$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda - \sigma & \sigma & 0 \\ r & -\lambda - 1 & 0 \\ 0 & 0 & -\lambda - \beta \end{vmatrix} \\ &= -(\lambda + \beta)\left[\lambda^2 + (\sigma + 1)\lambda + (1 - r)\sigma\right]. \end{aligned}$$

  Hence, the eigenvalues of A are

  $$\lambda = -\beta, \qquad \lambda = \frac{1}{2}\left[-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4(1 - r)\sigma}\right].$$

- We assume that the parameters σ, r, β are positive. Then $\lambda = -\beta < 0$ is a stable eigenvalue.

- Suppose that $r < 1$.

  - If $4(1-r)\sigma > (\sigma+1)^2$, then the remaining eigenvalues

$$\lambda = -\frac{1}{2}\left[(\sigma+1) \pm i\alpha\right], \qquad \alpha = \sqrt{4\sigma(1-r) - (\sigma+1)^2}$$

    are complex with negative real part.

  - If $0 \leq 4(1-r)\sigma < (\sigma+1)^2$, then

$$0 \leq \sqrt{(\sigma+1)^2 - 4(1-r)\sigma} < \sigma + 1,$$

    and the remaining eigenvalues are real and negative.

  In either case, the equilibrium $(0,0,0)^T$ is stable.

- Suppose that $r > 1$. Then

$$\sqrt{(\sigma+1)^2 - 4(1-r)\sigma} > \sigma + 1$$

  and therefore

$$\lambda = \frac{1}{2}\left[-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4(1-r)\sigma}\right] > 0$$

  so the equilibrium $(0,0,0)^T$ is unstable.

Remark. In the context of the Lorenz equation as a model of a fluid layer heated from below, this result has the interpretation that when the temperature difference (proportional to the Rayleigh number r) is sufficiently small, then a stationary equilibrium in which the fluid is a rest and transfers heat from bottom to top by conduction is stable. But when the temperature difference is too large, this equilibrium becomes unstable, leading to a convective motion.