Problem set 2
Math 207A, Fall 2011
Solutions

1. Solve the IVP for the logistic equation

$$x_t = x(1-x), \qquad x(0) = x_0.$$

Solution

- Separation of variables gives

  $$\int \frac{dx}{x(1-x)} = \int dt$$

  Using the partial fractions decomposition

  $$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

  to evaluate the x-integral, we get

  $$\log\left|\frac{x}{1-x}\right| = t + C,$$

  which implies that

  $$\frac{x}{1-x} = Ae^t$$

  where $A = \pm e^C$ is a constant of integration.

- Setting $t = 0$ and using the initial condition, we get

  $$A = \frac{x_0}{1-x_0}$$

  provided that $x_0 \neq 1$, and then the solution for x is

  $$x(t) = \frac{x_0 e^t}{1 - x_0 + x_0 e^t}.$$

  This also gives the correct solution $x(t) = 1$ if $x_0 = 1$.

2. Consider bacterial growth in a closed flask with a fixed initial amount of nutrient, and suppose that the growth rate of the bacteria is proportional to the amount of available nutrient. If $N(t)$ denotes the population of bacteria and $C(t)$ denotes the available nutrient at time t, explain why the ODEs

$$N_t = \mu CN, \qquad C_t = -\alpha\mu CN$$

provide a reasonable model for suitable constants $\alpha, \mu > 0$. Solve the system subject to the initial conditions

$$N(0) = N_0, \qquad C(0) = C_0$$

where $N_0, C_0 > 0$. Express the limiting population of bacteria

$$N_\infty = \lim_{t\to\infty} N(t)$$

in terms of α, μ, N_0, C_0. Does your answer make sense?

Solution

- The equation
$$N_t = \mu CN \tag{1}$$
states that the growth rate μC of bacteria is proportional to the nutrient amount C.

- We assume that the nutrient is consumed at a rate proportional to the growth rate of the bacteria (neglecting any uptake of nutrient that might be required to sustain the population at a constant level). This gives
$$C_t = -\alpha N_t \tag{2}$$
where $\alpha > 0$ is a constant of proportionality. Use of the equation for N_t gives
$$C_t = -\alpha\mu CN.$$

- We solve this system by eliminating C. From (2), we have
$$\frac{d}{dt}(C + \alpha N) = 0,$$
which implies that
$$C + \alpha N = B_0 \tag{3}$$

where B_0 is a constant of integration. From the initial condition,

$$B_0 = C_0 + \alpha N_0.$$

Equation (3) states that any decrease in the nutrient amount C is compensated by a proportional increase in the bacterial population N.

- Using (3) to express C in terms of N in (1), we get

$$N_t = \mu N \left(B_0 - \alpha N \right).$$

Thus, this model of population growth with limited resources leads to a logistic equation for N.

- Solving the logistic equation as in Problem 1, we get

$$N(t) = \frac{N_0 B_0 e^{\mu B_0 t}}{B_0 - \alpha N_0 + \alpha N_0 e^{\mu B_0 t}}.$$

- If $N_0 > 0$, then as $t \to \infty$ the solution approaches the limiting population $N_\infty = B_0/\alpha$ or

$$N_\infty = N_0 + \frac{C_0}{\alpha}$$

i.e. the initial population plus the population increase obtained by consumption of all of the nutrient.

3. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Find the equilibria of the ODE $x_t = f(x)$ and determine their stability, and sketch the phase line.

Solution

- The equilibria are

  $$x = 0, \qquad x = \frac{1}{n\pi}$$

  where $n \in \mathbb{Z}$ is any nonzero integer. There is an infinite sequence of equilibria $x = 1/(n\pi)$ that approach the equilibrium $x = 0$ as $n \to \infty$.

- For $x \neq 0$, we have

  $$f'(x) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right).$$

  Thus,

  $$f'\left(\frac{1}{n\pi}\right) = -\cos(n\pi) = (-1)^{n+1}.$$

  It follows that the equilibrium $x = 1/(n\pi)$ is asymptotically stable if n is even, when $f'(1/(n\pi) < 0$, and unstable if n is odd, when $f'(1/(n\pi) > 0$.

- The function f is differentiable, but not continuously differentiable, at the origin with

  $$f'(0) = \lim_{h \to 0} \left[\frac{h^2 \sin(1/h) - 0}{h}\right] = 0,$$

  so we cannot tell the stability of $x = 0$ from the sign of the derivative.

- Instead, note that if the initial condition is perturbed slightly from 0, the solution will approach the closest stable equilibrium to the initial data, but will not move further away from the origin. Thus the equilibrium $x = 0$ is stable but not asymptotically stable.

4. Graph the bifurcation diagram for equilibrium solutions of the scalar ODE

$$x_t = \mu + x - x^3$$

versus μ and determine their stability. (You don't have to give an explicit expression for the equilibria.) Find the values of (x, μ) at which equilibrium bifurcations occur. What kind of bifurcations are they? Sketch the phase line of the system for different values of μ, including the values at which bifurcations occur. Describe what would happen if the system is in equilibrium and μ is increased very slowly from $\mu = -1$ to $\mu = 1$ and then decreased back to $\mu = -1$.

Solution

- There is a cubic curve of equilibria as a function of μ,

  $$f(x; \mu) = 0, \qquad f(x; \mu) = \mu + x - x^3.$$

  The necessary condition $f_x = 0$ for an equilibrium bifurcation is

  $$1 - 3x^2 = 0.$$

  So the possible bifurcation points are

  $$(x, \mu) = \left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right), \quad \left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right).$$

- Sketching f shows that there is a unique globally asymptotically stable equilibrium if $|\mu| > 2/(3\sqrt{3})$, and three equilibria if $|\mu| < 2/(3\sqrt{3})$. The 'outside' ones are asymptotically stable and the middle one is unstable.

- Writing

  $$x = -\frac{1}{\sqrt{3}} + x_1 + \ldots, \qquad \mu = \frac{2}{3\sqrt{3}} + \mu_1 + \ldots$$

  we find that the Taylor expansion of f about the bifurcation point is

  $$f(x; \mu) = \mu_1 + \sqrt{3}x_1^2 + \ldots.$$

  Thus, there is a subcritical saddle-node bifurcation at

  $$(x, \mu) = \left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right).$$

Similarly, there is a supercritical saddle-node bifurcation at

$$(x, \mu) = \left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}} \right).$$

- When μ is increased quasi-statically from $\mu = -1$, the system will stay on the lower branch of stable equilibria until it is destroyed by the saddle-node bifurcation at $\mu = 2/(3\sqrt{3})$. The system will then jump to the upper stable equilibrium and move along the upper branch as μ is increased further. When μ is decreased from $\mu = 1$, the system will stay on the upper stable branch until it is destroyed by the saddle-node bifurcation at $\mu = -2/(3\sqrt{3})$. It will then jump back down to the lower stable equilibrium and stay on that branch as μ is decreased further.

- Note that reversing the changes in μ do not reverse the changes in the system, a phenomenon that is referred to as hysteresis.