PROBLEM SET 3

Math 207A, Fall 2011

Due: Wed., Oct. 19

1. For each of the the following systems, find the equilibria and their stability. Determine what bifurcations occurs, sketch the bifurcation diagram, and sketch the qualitatively different phase lines:

(a) $x_t = \mu - x^2 + x^4$; (b) $x_t = \mu x + x^3 - x^5$; (c) $x_t = \mu x - e^x$.

2. (a) Consider a pair of rigid rods of length L connected by a torsional spring with spring constant k that resists bending. If the rods are subject to a compressive force λ, and x is the angle of the rods to the applied force, explain why

$$V(x) = \frac{1}{2}kx^2 + 2\lambda L(\cos x - 1)$$

is a reasonable expression for the potential energy of the system.

(b) Show that equilibrium solutions such that $V'(x) = 0$ satisfy the equation

$$x - \mu \sin x = 0$$

where $\mu > 0$ is a suitable dimensionless parameter. Find and classify the bifurcation point on the branch $x = 0$ and give a physical interpretation. Sketch the behavior of the potential $V(x)$ as μ passes through the bifurcation value.

3. (a) A model of a fishery with harvesting is

$$N_t = \mu N\left(1 - \frac{N}{K}\right) - \frac{HN}{A + N}$$

where $N(t)$ is the population of fish at time t and μ, K, H, A are positive parameters. Explain why this is a reasonable model and give a biological interpretation of each of the parameters.

(b) Show that the ODE can be put in dimensionless form

$$x_t = x(1 - x) - \frac{hx}{a + x}$$

where t is a suitably rescaled time and a, h are dimensionless parameters. Give expressions for a, h in terms of the original dimensional parameters.

(c) Carry out a bifurcation analysis of the ODE in (b). Discuss the implications of your results for the original fish-harvesting problem.