Problem set 3
Math 207A, Fall 2011
Solutions

**1.** For each of the the following systems, find the equilibria and their stability. Determine what bifurcations occurs, sketch the bifurcation diagram, and sketch the qualitatively different phase lines:

(a) $x_t = \mu - x^2 + x^4$; (b) $x_t = \mu x + x^3 - x^5$; (c) $x_t = \mu x - e^x$.

**Solution.**

- (a) There is a subcritical saddle-node bifurcation at $(x, \mu) = (0, 0)$ and supercritical saddle-node bifurcations at $(x, \mu) = (\pm 1/\sqrt{2}, 1/4)$.
- (b) There is a subcritical pitchfork bifurcation at $(x, \mu) = (0, 0)$ and supercritical saddle-node bifurcations at $(x, \mu) = (\pm 1/\sqrt{2}, 1/4)$.
- (c) See Example 2.12 in the notes.

See the linked pdf file for the sketches.

**2.** (a) Consider a pair of rigid rods of length $L$ connected by a torsional spring with spring constant $k$ that resists bending. If the rods are subject to a compressive force $\lambda$, and $x$ is the angle of the rods to the applied force, explain why

$$V(x) = \frac{1}{2}kx^2 + 2\lambda L(\cos x - 1)$$

is a reasonable expression for the potential energy of the system.
(b) Show that equilibrium solutions such that $V'(x) = 0$ satisfy the equation

$$x - \mu \sin x = 0$$

where $\mu > 0$ is a suitable dimensionless parameter. Find and classify the bifurcation point on the branch $x = 0$ and give a physical interpretation. Sketch the behavior of the potential $V(x)$ as $\mu$ passes through the bifurcation value.

**Solution**

- (a) The potential energy of the spring is

  $$V_{\text{spring}}(x) = \frac{1}{2}kx^2$$

  and the work done on the rods by the external force (Force $\times$ Distance) is

  $$V_{\text{force}}(x) = 2\lambda L(1 - \cos x).$$

  Then

  $$V(x) = V_{\text{spring}}(x) - V_{\text{force}}(x).$$

- (b) We have

  $$V'(x) = kx - 2\lambda L \sin x = k\left(x - \mu \sin x\right), \qquad \mu = \frac{2\lambda L}{k}.$$

  So if $V'(x) = 0$, then $x = \mu \sin x$. Note that $k$ has dimensions of energy (the angle $x$ is dimensionless) as does $\lambda L$, so $\mu$ is dimensionless.

- The necessary condition for a bifurcation point is

  $$\frac{d}{dx}\left(x - \mu \sin x\right) = 1 - \mu \cos x = 0.$$

  So an equilibrium bifurcation off the branch $x = 0$ can occur only if $\mu = 1$.

- To Taylor expand $(x, \mu)$ about $(0, 1)$ we write

$$x = x_1 + \ldots, \qquad \mu = 1 + \mu_1 + \ldots.$$

Then

$$\begin{aligned} x - \mu \sin x &= x_1 - (1 + \mu_1) \sin x_1 + \ldots \\ &= x_1 - (1 + \mu_1) \left( x_1 - \frac{1}{6} x_1^3 + \ldots \right) \\ &= -\mu_1 x_1 + \frac{1}{6} x_1^3 + \ldots. \end{aligned}$$

Thus, near the bifurcation point we have to leading order that

$$\mu_1 x_1 = \frac{1}{6} x_1^3$$

corresponding to a supercritical pitchfork bifurcation.

- As $\mu$ passes through zero, the potential $V(x)$ changes from one with a single minimum at $x = 0$ to a bistable potential with two minima on either side of the origin and a maximum at $x = 0$.

**3.** (a) A model of a fishery with harvesting is

$$N_t = \mu N\left(1 - \frac{N}{K}\right) - \frac{HN}{A+N}$$

where $N(t)$ is the population of fish at time $t$ and $\mu$, $K$, $H$, $A$ are positive parameters. Explain why this is a reasonable model and give a biological interpretation of each of the parameters.

(b) Show that the ODE can be put in dimensionless form

$$x_t = x(1-x) - \frac{hx}{a+x}$$

where $t$ is a suitably rescaled time and $a$, $h$ are dimensionless parameters. Give expressions for $a$, $h$ in terms of the original dimensional parameters.

(c) Carry out a bifurcation analysis of the ODE in (b). Discuss the implications of your results for the original fish-harvesting problem.

**Solution**

- (a)The ODE has the form

$$N_t = \mu N\left(1 - \frac{N}{K}\right) - f(N), \qquad f(N) = \frac{HN}{A+N}.$$

- The first term $\mu N(1 - N/K)$ describes logistic growth of the fish population in the absence of harvesting, where $\mu$ is the maximum growth rate and $K$ is the carrying capacity of the system.

- The second term $f(N)$ gives the rate at which fish is harvested as a function of the population $N$. Note that $f(N)$ is a monotone increasing function of $N$. For populations $N$ that are much less than $A$, the harvesting rate $f(N) \sim \nu N$ is proportional to the available catch $N$, with rate constant

$$\nu = \frac{H}{A}.$$

  For populations much greater than $A$, the harvesting rate $f(N)$ approaches $H$. Thus, $H$ is the maximum rate at which fish is harvested, even if an unlimited catch is available, while $A$ is a measure of the fish population at which fish is harvested at a rate that is of the same order of the maximum rate *e.g.* $f(A) = H/2$.

- Note that if $P$ denotes a unit of population (*e.g.* tonnes of fish) and $T$ denotes a unit of time (*e.g.* years), then the parameters have dimensions

$$[\mu] = \frac{1}{T}, \qquad [K] = P, \qquad [H] = \frac{P}{T}, \qquad [A] = P.$$

- (b) Define dimensionless variable based on the logistic growth parameters,

$$x = \frac{N}{K}, \qquad \tilde{t} = \mu t.$$

Then

$$x_{\tilde{t}} = x(1-x) - \frac{hx}{a+x}, \qquad h = \frac{H}{\mu K}, \quad a = \frac{A}{K}.$$

Note that $h$, $a$ are dimensionless.

- (c) The equilibria satisfy

$$F(x; a, h) = 0, \qquad F(x; a, h) = x(1-x) - \frac{hx}{a+x}.$$

One solution, for all values of $a$, $h$, is

$$x = 0.$$

The other solutions are

$$x = \bar{x}_{\pm}, \qquad \bar{x}_{\pm} = \frac{1}{2}\left\{1 - a \pm \sqrt{(a+1)^2 - 4h}\right\}.$$

- There is a saddle-node bifurcation at $4h = (a+1)^2$, with two solutions appearing for

$$h < \frac{1}{4}(a+1)^2$$

These solutions appear at positive values if $a < 1$ and at a negative values if $a > 1$ (which are not relevant to the population problem).

- If $a \neq 1$, one of the solution branches $\bar{x}_{\pm}$ crosses the branch $x = 0$ at $h = a$, and there is a transcritical bifurcation at this point.

- If $h, a = 1$, then the saddle-node bifurcation and the transcritical bifurcation points coincide. Both branches $x_{\pm}$ appear from $x = 0$ at $(x, h, a) = (0, 1, 1)$ as one crosses the line $h = a$ transversely in the direction of increasing $a$, and there is a pitchfork bifurcation at that point.

- We have
$$F_x(x; a, h) = 1 - 2x - \frac{ha}{(a+x)^2}.$$

- The equilibrium $x = 0$ is stable if $F_x(0; a, h) < 0$ or $h/a > 1$ and unstable if $h/a < 1$. Note that
$$\frac{h}{a} = \frac{\nu}{\mu}$$
is the ratio of the harvesting rate constant to the growth constant. Thus $x = 0$ is stable if the per-capita harvesting is greater that the maximum growth rate, in which case a low fish population cannot be sustained.

- See the linked pdf file for sketches of the various phase lines as a function of the parameters $(a, h)$.