Problem set 4
Math 207A, Fall 2011
Solutions

1. Newton's method for the iterative solution of the scalar equation $f(x) = 0$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $f(x) = x^2 - 2$, show that this equation becomes

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

What are the fixed points of this system? Determine their stability. Compute x_4 numerically if $x_0 = 3$.

Solution

- For $f(x) = x^2 - 2$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

- Fixed points x satisfy

$$x = F(x), \qquad F(x) = \frac{x}{2} + \frac{1}{x},$$

  which implies that $x^2 = 2$ or $x = \pm\sqrt{2}$.

- We have

$$F'(x) = \frac{1}{2} - \frac{1}{x^2},$$

  so $F'(\pm 1/\sqrt{2}) = 0$. Since this has absolute value less than one, both fixed points are asymptotically stable. In fact, since $F' = 0$ at the fixed points, we get very rapid, quadratically nonlinear convergence of nearby iterates to the fixed point.

- We have

$$\begin{aligned}
F(3) &= 1.833333\ldots, \\
F(1.833333\ldots) &= 1.462121\ldots, \\
F(1.462121\ldots) &= 1.414998\ldots, \\
F(1.414998\ldots) &= 1.41421378\ldots.
\end{aligned}$$

  The forth iterate already agrees to six decimal places with $\sqrt{2} = 1.41421356\ldots$.

2. Find the fixed points of the system

$$x_{n+1} = -\frac{\mu}{2} \tan^{-1} x_n$$

and determine their stability. Show that a period-doubling bifurcation occurs at $\mu = 2$. Is the resulting period-two orbit stable or unstable?

Solution

- The fixed points satisfy

  $$x = -\frac{\mu}{2} \tan^{-1} x$$

  One solution branch is $x = 0$, defined for all values of μ. We can't solve for the other fixed points explicitly, but by looking at the intersection of the graphs

  $$y = -\frac{2}{\mu} x, \qquad y = \tan^{-1} x$$

  we see that there are two other fixed points $x = \pm\bar{x}(\mu)$ for $\mu < -2$, where $\bar{x}(\mu) \to \pi/2$ as $\mu \to -\infty$. For $\mu \geq -2$, $x = 0$ is the only fixed point.

- Writing the equation as $x_{n+1} = f(x_n; \mu)$ where

  $$f(x; \mu) = -\frac{\mu}{2} \tan^{-1} x,$$

  we have

  $$f_x(x; \mu) = -\frac{\mu}{2} \frac{1}{1 + x^2}.$$

- We have

  $$f_x(0; \mu) = -\frac{\mu}{2}$$

  so $x = 0$ is asymptotically stable for $|\mu| < 2$ and unstable if $|\mu| > 2$.

- To determine the stability of the other fixed points (which have the same stability since f is an odd function of x) note that

  $$f_x\left(\bar{x}(\mu); \mu\right) \to \infty \qquad \text{as } \mu \to -\infty.$$

  One can check that $f(\bar{x}, \mu) \neq 1$ for $\mu < -2$, so $f_x(\bar{x}, \mu) > 1$ by continuity. The fixed points are therefore unstable.

- The multiplier $f_x(0;\mu)$ passes through 1 at $\mu = -2$, and there is a subcritical pitchfork bifurcation of fixed points at $(x,\mu) = (0,-2)$, as shown by the previous solution for the fixed points of f.

- To determine the local behavior of the bifurcation at $(x,\mu) = (0,2)$, we Taylor expand about this point. Writing $\mu = 2 + \mu_1$, we get

$$\begin{aligned} f(x,\mu) &= -\left(1 + \frac{1}{2}\mu_1\right)\left(x - \frac{1}{3}x^3 + \dots\right) \\ &= -\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \dots \end{aligned}$$

  and

$$\begin{aligned} f^2(x,\mu) &= -\left(1 + \frac{1}{2}\mu_1\right)\left[-\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \dots\right] \\ &\quad + \frac{1}{3}\left[-\left(1 + \frac{1}{2}\mu_1\right)x + \frac{1}{3}x^3 + \dots\right]^3 + \dots \\ &= (1 + \mu_1)\,x - \frac{2}{3}x^3 + \dots \end{aligned}$$

- Thus f^2 has a supercritical pitchfork bifurcation at $(0,2)$, and f has a supercritical period-doubling bifurcation in which a stable periodic orbit appears above $\mu = 2$. The points on the orbit are fixed points of f^2; they are given approximately by

$$x = \pm\sqrt{\frac{3(\mu - 2)}{2}} + \dots .$$

3. Consider the discrete dynamical system on the circle for $x_n \in \mathbb{T}$

$$x_{n+1} = x_n + \mu \qquad (\text{mod } 2\pi)$$

corresponding to rotation by an angle $\mu \in \mathbb{T}$. Describe the structure of the orbits and how they depend on μ.

Solution

- The orbit structure depends on whether μ is a rational or irrational multiple of 2π.
- If
$$\frac{\mu}{2\pi} = \frac{p}{q} \in \mathbb{Q}$$
where p, q are relatively prime, then every orbit is periodic with minimal period q.
- If
$$\frac{\mu}{2\pi} \notin \mathbb{Q}$$
then every orbit consists of a countably infinite sequence of distinct points. One can show that every orbit is dense in $\mathbb{T}$ and equidistributed (Weyl's theorem).

4. Carry out numerical experiments for iterations of the logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

where $1 \leq \mu \leq 4$ and $0 \leq x_0 \leq 1$. (You can write your own program or use the MATLAB script provided on the course website.)

Solution

- You should see a sequence of period doubling bifurcations, followed by chaotic behavior. Inside the chaotic region, there are 'windows' with stable period 3 orbits. See the text for further details.