PROBLEM SET 5
Math 207A, Fall 2011
Due: Wed., Nov. 2

1. Solve the ODE

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

If $\omega > 0$, sketch the phase plane for $\mu > 0$, $\mu = 0$, and $\mu < 0$. Classify the equilibrium $\vec{x} = 0$ in each case.

2. Solve the ODE

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Sketch the phase plane for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Classify the equilibrium $\vec{x} = 0$ in each case.

3. (a) Use the power series definition of the exponential of a linear map

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

to show that if $AB = BA$ then $e^{tA}e^{tB} = e^{t(A+B)}$. (You can assume that the series can be multiplied term by term and rearranged.)
(b) If

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

use the series definition and the result of (a) to compute e^{tA}. Compare your answer with the solution of Problem 2.
(c) If A, B need not commute, show that as $t \to 0$

$$e^{tA}e^{tB} = e^{t(A+B)} + \frac{1}{2}t^2[A, B] + O(t^3)$$

where $[A, B] = AB - BA$ is the commutator of A and B.

4. Let $\vec{f} : \mathbb{R} \to \mathbb{R}^d$ be a continuous vector-valued function and $A : \mathbb{R}^d \to \mathbb{R}^d$ a linear map. Show that the solution $\vec{x} : \mathbb{R} \to \mathbb{R}^d$ of the autonomous, nonhomogeneous system

$$\vec{x}_t = A\vec{x} + \vec{f}(t), \qquad \vec{x}(0) = \vec{x}_0$$

is given by

$$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\, ds.$$

(This expression for the solution of the nonhomogeneous equation in terms of the solution of the homogeneous equation is called Duhamel's formula.)