Problem set 5
Math 207A, Fall 2011
Solutions

1. Solve the ODE

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

If $\omega > 0$, sketch the phase plane for $\mu > 0$, $\mu = 0$, and $\mu < 0$. Classify the equilibrium $\vec{x} = 0$ in each case.

Solution

- The characteristic polynomial of the matrix is

$$\begin{vmatrix} \mu - \lambda & -\omega \\ \omega & \mu - \lambda \end{vmatrix} = (\lambda - \mu)^2 + \omega^2.$$

- The eigenvalues (roots of this polynomial) are

$$\lambda_\pm = \mu \pm i\omega.$$

  Thus, the origin is a stable spiral if $\mu < 0$, a center if $\mu = 0$, and an unstable spiral if $\mu > 0$. For $\omega > 0$, the trajectories rotate counter-clockwise since $x_t < 0$ when $x = 0$ and $y > 0$.

- Eigenvectors corresponding to $\lambda_\pm$ are

$$\vec{r}_\pm = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}.$$

  Thus, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(\mu + i\omega)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(\mu - i\omega)t} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

- The initial conditions are satisfied if

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix},$$

  which implies that

$$c_1 = \frac{1}{2}\left(x_0 + iy_0\right), \qquad c_2 = \frac{1}{2}\left(x_0 - iy_0\right).$$

- Writing this solution in real form, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\mu t} \begin{pmatrix} x_0 \cos \omega t - y_0 \sin \omega t \\ x_0 \sin \omega t + y_0 \cos \omega t \end{pmatrix}.$$

- Alternatively, note that

$$\begin{aligned} \exp t \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} &= \exp t \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \cdot \exp t \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\mu t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \cdot \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \cos \omega t & \sin \omega t \end{pmatrix} \end{aligned}$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \cos \omega t & \sin \omega t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

which gives the same solution.

2. Solve the ODE

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Sketch the phase plane for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Classify the equilibrium $\vec{x} = 0$ in each case.

Solution

- We have

  $$y_t = \lambda y, \qquad y(0) = y_0$$

  so

  $$y(t) = y_0 e^{\lambda t}.$$

- Then

  $$x_t = \lambda x + y_0 e^{\lambda t}, \qquad x(0) = x_0$$

  so

  $$x(t) = x_0 e^{\lambda t} + y_0 t e^{\lambda t}.$$

- Thus

  $$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

- If $\lambda < 0$ the origin is a stable node, if $\lambda > 0$ it is an unstable node, and if $\lambda = 0$ it is singular (the x-axis consists of equilibria).

3. (a) Use the power series definition of the exponential of a linear map

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

to show that if $AB = BA$ then $e^{tA}e^{tB} = e^{t(A+B)}$. (You can assume that the series can be multiplied term by term and rearranged.)

(b) If

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

use the series definition and the result of (a) to compute e^{tA}. Compare your answer with the solution of Problem 2.

(c) If A, B need not commute, show that as $t \to 0$

$$e^{tA}e^{tB} = e^{t(A+B)} + \frac{1}{2}t^2[A, B] + O(t^3)$$

where $[A, B] = AB - BA$ is the commutator of A and B.

Solution

- (a) The proof is the same as in the scalar case. We have

$$\begin{aligned} e^{tA}e^{tB} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} t^m B^m \right) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!m!} t^{k+m} A^k B^m \\ &= \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!} A^k B^{n-k} \right). \end{aligned}$$

  If A, B commute, then

$$(A + B)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k}.$$

  Hence

$$\begin{aligned} e^{tA}e^{tB} &= \sum_{n=0}^{\infty} \frac{1}{n!} t^n (A + B)^n \\ &= e^{t(A+B)}. \end{aligned}$$

The term-by-term multiplication and rearrangement of the series is justified by their absolute convergence.

- (b) Writing A as

$$A = \Lambda + N, \qquad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have

$$\Lambda^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^n \end{pmatrix}$$

so

$$e^{t\Lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \Lambda^n = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \lambda^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} t^n \lambda^n \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Also $N^n = 0$ for $n \geq 2$, so

$$e^{tN} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- The matrices Λ, N commute so by (a)

$$e^{tA} = e^{t\Lambda} e^{tN} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- (c) If A and B do not commute, then

$$\begin{aligned} e^{tA} e^{tB} &= \left(1 + tA + \frac{1}{2} t^2 A^2\right)\left(1 + tB + \frac{1}{2} t^2 B^2\right) + O(t^3) \\ &= 1 + t(A+B) + \frac{1}{2} t^2 \left(A^2 + 2AB + B^2\right) + O(t^3) \end{aligned}$$

and

$$\begin{aligned} e^{t(A+B)} &= 1 + t(A+B) + \frac{1}{2} t^2 (A+B)^2 + O(t^3) \\ &= 1 + t(A+B) + \frac{1}{2} t^2 \left(A^2 + AB + BA + B^2\right) + O(t^3). \end{aligned}$$

- Subtracting these expressions, we get

$$e^{tA} e^{tB} - e^{t(A+B)} = \frac{1}{2} t^2 \left(AB - BA\right) + O(t^3)$$

which proves the result.

4. Let $\vec{f} : \mathbb{R} \to \mathbb{R}^d$ be a continuous vector-valued function and $A : \mathbb{R}^d \to \mathbb{R}^d$ a linear map. Show that the solution $\vec{x} : \mathbb{R} \to \mathbb{R}^d$ of the autonomous, nonhomogeneous system

$$\vec{x}_t = A\vec{x} + \vec{f}(t), \qquad \vec{x}(0) = \vec{x}_0$$

is given by

$$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\, ds.$$

(This expression for the solution of the nonhomogeneous equation in terms of the solution of the homogeneous equation is called Duhamel's formula.)

Solution

- Multiplying the equation by e^{-tA}, we get

  $$e^{-tA}\left(\vec{x}_t - A\vec{x}\right) = e^{-tA}\vec{f}(t).$$

  The left-hand side is an exact time-derivative, so

  $$\frac{d}{dt}\left(e^{-tA}\vec{x}\right) = e^{-tA}\vec{f}(t).$$

- Integrating this equation with respect to t and imposing the initial condition, we get

  $$e^{-tA}\vec{x}(t) = \vec{x}_0 + \int_0^t e^{-sA}\vec{f}(s)\, ds.$$

  Multiplication of this equation by e^{tA} gives

  $$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\, ds.$$

Remark. As follows from the superposition property of linear equations, the solution is the sum of a solution of the homogeneous equation $\vec{x}_t = A\vec{x}$ with initial data $\vec{x}(0) = \vec{x}_0$, and a solution of the nonhomogeneous equation $\vec{x}_t = A\vec{x} + \vec{f}$ with zero initial data $\vec{x}(0) = 0$.