Problem set 6
Math 207A, Fall 2011
Due: Wed., Nov. 9

1. A two-dimensional gradient system has the form

$$x_t = -\frac{\partial W}{\partial x}(x, y), \qquad y_t = -\frac{\partial W}{\partial y}(x, y)$$

where $W(x, y)$ is a given function.

(a) If W is a quadratic function

$$W(x, y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

show that this is a linear system and classify the possible types of equilibria that can arise. How are the trajectories related to the level curves of W?

(b) For what values of a, b, c does the flow decrease, increase, or leave unchanged areas in the phase plane?

(c) Sketch the phase plane if: (i) $W(x, y) = x^2 + 4y^2$; (ii) $W(x, y) = xy$.

2. A two-dimensional Hamiltonian system has the form

$$x_t = \frac{\partial H}{\partial y}(x, y), \qquad y_t = -\frac{\partial H}{\partial x}(x, y)$$

where $H(x, y)$ is a given function.

(a) If H is a quadratic function

$$H(x, y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

show that this is a linear system and classify the possible types of equilibria that can arise. How are the trajectories related to the level curves of H?

(b) For what values of a, b, c does the flow decrease, increase, or leave unchanged areas in the phase plane?

(c) Sketch the phase plane if: (i) $H(x, y) = x^2 + 4y^2$; (ii) $H(x, y) = xy$.

3. Consider an IVP for a 2×2 non-autonomous, homogeneous system

$$\vec{x}_t = A(t)\vec{x}, \qquad \vec{x}(t_0) = \vec{x}_0, \tag{1}$$

where the 2×2 matrix $A(t)$ depends on time t. For $k = 1, 2$, let $\vec{x}_k(t)$ be the solution of (1) with $\vec{x}_k(t_0) = \vec{e}_k$, where $\vec{e}_1 = (1,0)^T$, $\vec{e}_2 = (0,1)^T$ are the standard basis vectors of $\mathbb{R}^2$. Show that the solution of (1) is

$$\vec{x}(t) = X(t; t_0)\vec{x}_0$$

where the 2×2 matrix

$$X = [\vec{x}_1, \vec{x}_2]$$

has columns $\vec{x}_k$, $k = 1, 2$. (The matrix X plays the role of the solution matrix e^{tA} in the autonomous case, although we typically cannot compute it explicitly.)

4. (a) Consider the one-dimensional motion of an undamped particle of unit mass in a bistable potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$

The position $x(t)$ of the particle satisfies the ODE

$$x_{tt} - x + x^3 = 0.$$

Find the equilibria, classify them, and sketch the phase portrait in the (x, x_t)-plane.

(b) Repeat part (a) if the particle is linearly damped, so that

$$x_{tt} + \delta x_t - x + x^3 = 0$$

for some $\delta > 0$.