PROBLEM SET 6
Math 207A, Fall 2011
Solutions

1. A two-dimensional gradient system has the form

$$x_t = -\frac{\partial W}{\partial x}(x,y), \qquad y_t = -\frac{\partial W}{\partial y}(x,y)$$

where $W(x,y)$ is a given function.
(a) If W is a quadratic function

$$W(x,y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

show that this is a linear system and classify the possible types of equilibria that can arise. How are the trajectories related to the level curves of W?
(b) For what values of a, b, c does the flow decrease, increase, or leave unchanged areas in the phase plane?
(c) Sketch the phase plane if: (i) $W(x,y) = x^2 + 4y^2$; (ii) $W(x,y) = xy$.

Solution

- (a) The linear system is

  $$\begin{pmatrix} x \\ y \end{pmatrix}_t = -\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

  The matrix

  $$A = -\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

  is symmetric and therefore has real eigenvalues with orthogonal eigenvectors. The only equilibria that can arise are nodes, saddle points, and singular points. The trajectories are orthogonal to the level curves of W and in the direction of decreasing W.

- Explicitly, the characteristic equation is

  $$\begin{vmatrix} -\lambda - a & -b \\ -b & -\lambda - c \end{vmatrix} = \lambda^2 + (a+c)\lambda + ac - b^2 = 0$$

with roots

$$\begin{aligned}\lambda &= \frac{1}{2}\left[-(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}\right] \\ &= \frac{1}{2}\left[-(a+c) \pm \sqrt{(a-c)^2 + 4b^2}\right].\end{aligned}$$

The discriminant is positive, so the eigenvalues are real. Depending on the sign of the determinant of A, we get a saddle point if $ac < b^2$ (W is a hyperboloid), a singular point if $ac = b^2$, and a node if $ac > b^2$ (W is a paraboloid). The node is stable if $a+c > 0$ (upward opening paraboloid) and unstable if $a+c < 0$ (downward opening paraboloid).

- (b) We have
$$\det e^{tA} = e^{t \operatorname{tr} A} = e^{-(a+c)t}.$$
Thus, the flow increases areas if $a+c < 0$, preserves areas if $a+c = 0$ (corresponding to a saddle point whose flow contracts and expands by an equal factor in orthogonal directions), and decreases areas if $a+c > 0$.

- (c) In (i), the origin is a stable node, with trajectories orthogonal to the ellipses $x^2 + 4y^2 =$ constant. In (ii), the origin is a saddle point with trajectories orthogonal to the hyperbolas $xy =$ constant.

2. A two-dimensional Hamiltonian system has the form

$$x_t = \frac{\partial H}{\partial y}(x, y), \qquad y_t = -\frac{\partial H}{\partial x}(x, y)$$

where $H(x, y)$ is a given function.

(a) If H is a quadratic function

$$H(x, y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$$

show that this is a linear system and classify the possible types of equilibria that can arise. How are the trajectories related to the level curves of H?

(b) For what values of a, b, c does the flow decrease, increase, or leave unchanged areas in the phase plane?

(c) Sketch the phase plane if: (i) $H(x, y) = x^2 + 4y^2$; (ii) $H(x, y) = xy$.

Solution

- (a) The linear system is

  $$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  The Hamiltonian is constant on solutions, so the trajectories coincide with the level curves of H.

- The characteristic equation is

  $$\begin{vmatrix} -\lambda + b & c \\ -a & -\lambda - b \end{vmatrix} = \lambda^2 + ac - b^2 = 0$$

  with roots

  $$\lambda = \pm\sqrt{b^2 - ac}.$$

  We get a saddle point if $ac < b^2$ (H is a hyperboloid), a singular point if $ac = b^2$, and a center if $ac > b^2$ (H is a paraboloid).

- (b) We have

  $$\det e^{tA} = e^{t \operatorname{tr} A} = e^{(b-b)t} = 1.$$

  Thus, a Hamiltonian flow always preserves areas in phase space.

- (c) In (i), the origin is a nonlinear center, with elliptical orbits $x^2+4y^2 =$ constant. In (ii), the origin is a saddle point whose trajectories are the hyperbolas $xy =$ constant.

3. Consider an IVP for a 2×2 non-autonomous, homogeneous system

$$\vec{x}_t = A(t)\vec{x}, \qquad \vec{x}(t_0) = \vec{x}_0, \tag{1}$$

where the 2×2 matrix $A(t)$ depends on time t. For $k = 1, 2$, let $\vec{x}_k(t)$ be the solution of (1) with $\vec{x}_k(t_0) = \vec{e}_k$, where $\vec{e}_1 = (1, 0)^T$, $\vec{e}_2 = (0, 1)^T$ are the standard basis vectors of $\mathbb{R}^2$. Show that the solution of (1) is

$$\vec{x}(t) = X(t; t_0)\vec{x}_0$$

where the 2×2 matrix

$$X = [\vec{x}_1, \vec{x}_2]$$

has columns $\vec{x}_k$, $k = 1, 2$. (The matrix X plays the role of the solution matrix e^{tA} in the autonomous case, although we typically cannot compute it explicitly.)

Solution

- By linearity, the function

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = [\vec{x}_1(t), \vec{x}_2(t)] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

  is a solution of the ODE for any constants c_1, c_2.

- Imposing the initial condition, we find that

$$\vec{x}(t_0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

  or

$$\vec{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

  Defining $X = [\vec{x}_1, \vec{x}_2]$ we get the result.

4. (a) Consider the one-dimensional motion of an undamped particle of unit mass in a bistable potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$

The position $x(t)$ of the particle satisfies the ODE

$$x_{tt} - x + x^3 = 0.$$

Find the equilibria, classify them, and sketch the phase portrait in the (x, x_t)-plane.
(b) Repeat part (a) if the particle is linearly damped, so that

$$x_{tt} + \delta x_t - x + x^3 = 0$$

for some $\delta > 0$.

Solution

- (a) The equation of the trajectories is

$$\frac{1}{2}x_t^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = E$$

  where E is a constant. This gives a phase portrait like a "pair of spectacles."

- The equilibria are $x = 0$ (saddle with $V''(0) = -1 < 0$ and a local maximum in V) and $x = \pm 1$ (nonlinear centers with $V''(\pm 1) = 2 > 0$ and a local minimum in V). There are two homoclinic orbits that approach $x = 0$ as $t \to \pm\infty$ in which the left or right part of the unstable manifold of the saddle point loops around the equilibrium $x = \pm 1$ before returning to $x = 0$ along the left or right part of the stable manifold

- (b) When damping is included, the equilibrium $x = 0$ remains a saddle point, while the equilibria $x = \pm 1$ become stable spirals if $0 < \delta < \sqrt{8}$ (the underdamped case) or stable nodes if $\delta \geq \sqrt{8}$ (the overdamped case). The homoclinic orbits are broken by the inclusion of damping, and the left or right part of the unstable manifold of the saddle point at $x = 0$ approaches the corresponding left or right asymptotically stable equilibrium at $x = \pm 1$.