PROBLEM SET 7
Math 207A, Fall 2011
Solutions

1. Classify the equilibrium $(x, y) = (0, 0)$ of the system

$$x_t = x, \qquad y_t = -y + x^2.$$

Is the equilibrium hyperbolic? Find an equation for the trajectories in (x, y)-phase space, and sketch the phase plane. What are the stable and unstable subspaces E^s and E^u and the stable and unstable manifolds $W^s(0, 0)$ and $W^u(0, 0)$ of the origin?

Solution

- The linearized system at the origin is $x_t = x$, $y_t = -y$ or

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  The eigenvalues are $\lambda = \pm 1$, so the origin is a saddle point. This is hyperbolic since neither eigenvalue has zero real part.

- The equation of the trajectories is

$$\frac{dy}{dx} = \frac{y_t}{x_t} = \frac{-y + x^2}{x} = -\frac{1}{x} y + x.$$

  This is a linear equation for $y(x)$ with integrating factor x:

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x^2.$$

  The solution is

$$y = \frac{1}{3}x^2 + \frac{C}{x} \tag{1}$$

  where C is a constant of integration.

- For $\lambda = 1$, the eigenvector is $\vec{r} = (1, 0)^T$, which spans the unstable subspace

$$E^u = \left\{ \begin{pmatrix} c \\ 0 \end{pmatrix} : c \in \mathbb{R} \right\}$$

  i.e. the x-axis.

- For $\lambda = -1$, the eigenvector is $\vec{r} = (0, 1)^T$, which spans the stable subspace
$$E^s = \left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} : c \in \mathbb{R} \right\}$$
i.e. the y-axis.

- The y-axis is invariant and tangent to E^s at the origin, so the stable manifold $W^s(0, 0)$ is the y-axis, $x = 0$.

- The curve (1) with $C = 0$ is invariant and tangent to E^u at the origin, so the unstable manifold $W^u(0, 0)$ is the parabola
$$y = \frac{1}{3}x^2.$$

2. Write the system

$$
\begin{aligned}
x_t &= x - y - x\left(x^2 + y^2\right) \\
y_t &= x + y - y\left(x^2 + y^2\right)
\end{aligned}
$$

in polar coordinates. Classify the equilibrium $(x, y) = (0, 0)$ and sketch the phase portrait. How do solutions behave as $t \to \infty$?

Solution

- Writing

  $$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1} \frac{y}{x}$$

  we have

  $$
  \begin{aligned}
  r_t &= \frac{x x_t + y y_t}{r} = \frac{x(x - y - xr^2) + y(x + y - yr^2)}{r} = r - r^3, \\
  \theta_t &= \frac{x y_t - x y_t}{x^2 + y^2} = \frac{x(x + y - yr^2) - y(x - y - xr^2)}{r^2} = 1,
  \end{aligned}
  $$

  so the polar form of the ODE is

  $$r_t = r - r^3, \qquad \theta_t = 1.$$

- The origin $r = 0$ is an unstable spiral point and the circle $r = 1$ is a stable limit cycle.

- As $t \to \infty$ any solution with nonzero initial data approaches the limit cycle

  $$x(t) = \cos(t + \delta), \qquad y(t) = \sin(t + \delta)$$

  for some constant phase shift δ (which depends on the initial data).

3. Find and classify the equilibria of the system

$$x_t = \mu x - x^2, \qquad y_t = -y.$$

Sketch the phase portraits for $\mu < 0$, $\mu = 0$, and $\mu > 0$. In each case, say if the equilibria are hyperbolic and describe their stable and unstable subspaces E^s and E^u and their stable and unstable manifolds W^s and W^u.

Solution

- The equilibria are

$$(x, y) = (0, 0), \qquad (x, y) = (\mu, 0).$$

- The linearization at $(0, 0)$ is

$$x_t = \mu x, \qquad y_t = -y.$$

  If $\mu < 0$, this is a stable node (with $E^s = \mathbb{R}^2$, $E^u = 0$), if $\mu = 0$ this is a singular point (with E^s the y-axis and $E^u = 0$), and if $\mu > 0$ this is a saddle point (with E^s the y-axis and E^u the x-axis). The equilibrium is hyperbolic if $\mu \neq 0$.

- The linearization at $(\mu, 0)$ is

$$x_t = -\mu x, \qquad y_t = -y$$

  where (x, y) denote the perturbations from the equilibrium $(\mu, 0)$. If $\mu < 0$, this is a saddle point (with E^s the y-axis, E^u the x-axis), if $\mu = 0$ this is a singular point (with E^s the y-axis and $E^u = 0$), and if $\mu > 0$ this is a stable node (with $E^s = \mathbb{R}^2$ and $E^u = 0$). The equilibrium is hyperbolic if $\mu \neq 0$.

- Note the exchange of stability in the equilibria when they cross at $\mu = 0$.

- Looking at the phase planes, shown in the linked figure, we see that:

  - if $\mu < 0$ then

$$\begin{aligned} W^s(\mu, 0) &= \{(\mu, y) : y \in \mathbb{R}\}, \\ W^u(\mu, 0) &= \{(x, 0) : -\infty < x < 0\}, \\ W^s(0, 0) &= \{(x, y) : \mu < x < \infty, y \in \mathbb{R}\}, \\ W^u(0, 0) &= \{(0, 0)\}. \end{aligned}$$

- if $\mu = 0$ then

$$\begin{aligned} W^s(0,0) &= \{(0,y) : y \in \mathbb{R}\}\,, \\ W^u(0,0) &= \{(0,0)\}. \end{aligned}$$

  Note that the stable manifold only includes trajectories that approach $(0,0)$ tangent to the stable subspace, not ones that approach $(0,0)$ tangent to the center subspace (which do not approach the equilibrium exponentially quickly).

- if $\mu > 0$ then

$$\begin{aligned} W^s(0,0) &= \{(0,y) : y \in \mathbb{R}\}\,, \\ W^u(0,0) &= \{(x,0) : -\infty < x < \mu\}\,, \\ W^s(\mu,0) &= \{(x,y) : 0 < x < \infty, y \in \mathbb{R}\}\,, \\ W^u(\mu,0) &= \{(0,0)\}. \end{aligned}$$

4. Consider the following model for the dynamics of a predator with population $x(t)$ and a prey with population $y(t)$ *e.g.* pikes and eels, or foxes and rabbits:

$$x_t = x\,(-1 + y)\,,$$
$$y_t = y\,(1 - x)\,.$$

Explain why this is a reasonable qualitative model for a predator-prey system. Find the equilibria and classify them. Sketch the phase portrait. How do solutions behave?

Solution

- In the absence of prey ($y = 0$), the predator population satisfies

  $$x_t = -x$$

  so it decays exponentially; with a sufficiently large population of prey ($y > 1$), the predator population grows. In the absence of predators ($x = 0$), the prey population satisfies

  $$y_t = y$$

  so it grows exponentially; with a sufficiently large population of predators ($x > 1$), the prey population declines.

- The equilibria are

  $$(x, y) = (0, 0), \qquad (x, y) = (1, 1).$$

- The linearization at $(0, 0)$ is

  $$x_t = -x, \qquad y_t = y,$$

  which is a saddle point with eigenvalues $\lambda = \pm 1$.

- The linearization at $(1, 1)$ is

  $$x_t = y, \qquad y_t = -x$$

  where (x, y) denote the perturbations from the equilibrium $(1, 1)$. This is a center with eigenvalues $\lambda = \pm i$.

- The trajectories satisfy
$$\frac{dy}{dx} = \frac{y_t}{x_t} = \frac{x\,(-1+y)}{y\,(1-x)}.$$
Separating variables, we get that
$$\int \frac{-1+y}{y}\,dy = \int \frac{1-x}{x}\,dx$$
or
$$-\log|y| + y = \log|x| - x + C$$
where C is a constant of integration. Hence, taking the exponential of this equation and renaming the constant, we find that the trajectories satisfy
$$ye^{-y} = C\frac{e^x}{x}. \tag{2}$$

- Consider $x, y > 0$. The function
$$f(x) = \frac{e^x}{x}$$
is monotone decreasing in $0 < x \leq 1$ from ∞ to a minimum value $f(1) = e$ and monotone increasing in $1 \leq x < \infty$ from e to ∞. The function
$$g(y) = ye^{-y}$$
is monotone increasing in $0 < y \leq 1$ from 0 to a maximum value $g(1) = 1/e$, and monotone decreasing in $1 \leq y < \infty$ from $1/e$ to 0. It follows that (2) has positive solutions only if $C \geq 1/e^2$. When $C = 1/e^2$, the unique solution is the equilibrium trajectory $(x, y) = (1, 1)$. When $C > 1$, the equation defines a closed bounded curve enclosing the equilibrium.

- It follows that $(1, 1)$ is a nonlinear center for the full system and that all solutions with positive initial data are periodic functions of time.

- According to this model, if initially the predator population is small and the prey population is large, then the predator population increases at the expense of the prey until there are too many predators; the predator population then decreases and the prey population recovers. This cycle then repeats.