Problem set 8
Math 207A, Fall 2011
Solutions

1. Sketch the phase plane of the system

$$x_t = x^2, \qquad y_t = -y.$$

Linearize the system about the equilibrium $(0,0)$ and determine the unstable, stable and center subspaces of the equilibrium. What is the stable manifold $W^s(0,0)$? Show that there are many possible choices of a (C^∞) center manifold $W^c(0,0)$.

Solution

- the linearized system at $(0,0)$ is

$$x_t = 0, \qquad y_t = -y.$$

  This has eigenvalues $\lambda = -1, 0$ with corresponding eigenvectors $\vec{r} = (0,1)^T, (1,0)^T$ which span the (one-dimensional) stable and center subspaces, respectively. The (zero-dimensional) unstable subspace consists of 0.

- The equation of the trajectories is

$$\frac{dy}{dx} = -\frac{y}{x^2}$$

  Separating variables and solving this equation, we find that

$$y = Ce^{1/x}$$

  where C is a constant of integration.

- For $C \neq 0$, the trajectories approach the origin smoothly (C^∞) as $x \to 0^-$, go to infinity as $x \to 0^+$,a and approach the horizontal asymptote $y = C$ as $|x| \to \infty$. The phase portraits is the same as the one shown for question 3 in problem set 7 with $\mu = 0$ reflected in the y-axis $(x \mapsto -x)$.

- The stable subspace of the origin, the y-axis, is invariant under the flow, so it is also the stable manifold. The unstable manifold is 0.

- Any curve of the form

$$y = \begin{cases} Ce^{1/x} & -\infty < x < 0 \\ 0 & 0 \leq x \end{cases}$$

  for some constant C is a smooth (C^∞) invariant manifold. (It consists of three trajectories: the part for $x < 0$, the equilibrium 0, and the positive x-axis.) It is tangent to the center subspace at 0, so any such curve is a center manifold. In particular, the whole x-axis is a center manifold ($C = 0$), but it is not the only one.

2. Consider the Euler equations for a rotating rigid body

$$\dot{M}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_2 M_3,$$
$$\dot{M}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_3 M_1,$$
$$\dot{M}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_1 M_2,$$

where $M_1(t)$, $M_2(t)$, $M_3(t)$ are components of the body angular momentum and the positive constants $0 < I_1 < I_2 < I_3$ are the moments of inertia of the body (which we assume to be distinct).

(a) Show that the (squared) total angular momentum

$$J = M_1^2 + M_2^2 + M_3^2$$

and the kinetic energy

$$T = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}$$

are conserved.

(b) Restrict the Euler equations to the sphere

$$M_1^2 + M_2^2 + M_3^2 = 1, \tag{1}$$

which is an invariant manifold for the flow by (a). Find the equilibria on this sphere, linearize the equations about the equilibria, classify them, and determine their stability. Sketch the phase portrait on the sphere.

Solution

- (a) We have

$$\begin{aligned}\dot{J} &= M_1\dot{M}_1 + M_2\dot{M}_2 + M_3\dot{M}_3 \\ &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_1 M_2 M_3 + \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_2 M_3 M_1 \\ &\qquad + \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_3 M_1 M_2 \\ &= 0\end{aligned}$$

and

$$
\begin{aligned}
\dot{T} &= \frac{M_1}{I_1}\dot{M}_1 + \frac{M_2}{I_2}\dot{M}_2 + \frac{M_3}{I_3}\dot{M}_3 \\
&= \left(\frac{1}{I_3 I_1} - \frac{1}{I_1 I_2}\right) M_1 M_2 M_3 + \left(\frac{1}{I_1 I_2} - \frac{1}{I_2 I_3}\right) M_2 M_3 M_1 \\
&\qquad\qquad + \left(\frac{1}{I_2 I_3} - \frac{1}{I_3 I_1}\right) M_3 M_1 M_2 \\
&= 0
\end{aligned}
$$

so both J and T are conserved.

- (b) There are six equilibria on the unit sphere:

$$(M_1, M_2, M_3) = (\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1).$$

These correspond to a steady rotation of the body in either direction about each of its three principal axes with total angular momentum one.

- Consider the linearization about $(1, 0, 0)$. Writing

$$M_1 = 1 + N_1, \qquad M_2 = N_2, \qquad M_3 = N_3$$

where N_1, N_2, N_3 are small, we find from the constraint (1) that

$$2N_1 + N_2^2 + N_2^2 = 0.$$

Thus, in the linearized approximation, we have $N_1 = 0$. (The tangent plane to the sphere at $(1, 0, 0)$ is vertical.) This is consistent with the linearization of the first equation, which gives $\dot{N}_1 = 0$. Linearizing the remaining equations at $M_1 = 1$, we get

$$\dot{N}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) N_3, \qquad \dot{N}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) N_2.$$

Since $0 < I_1 < I_2 < I_3$, the coefficients $1/I_1 - 1/I_3$ and $1/I_2 - 1/I_1$ have opposite signs, so the equilibrium is a center. A similar computation shows that $(-1, 0, 0)$ and $(0, 0, \pm 1)$ are centers.

- Linearizing about $(0,1,0)$, we find that

$$\dot{N}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) N_3, \qquad \dot{N}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) N_1.$$

  In this case, both coefficients $1/I_3 - 1/I_2$ and $1/I_2 - 1/I_1$ have the same sign, so the equilibrium is a saddle. Similarly, $(0,-1,0)$ is a saddle.

- Thus, steady rotations about the principal axes with the largest and smallest moments of inertia are linearly stable, but a steady rotation about the axis with the middle moment of inertia is unstable.

- We can't immediately conclude from the previous analysis that the equilibria $(\pm 1, 0, 0)$, $(0, 0, \pm 1)$ are nonlinearly stable, since centers are not hyperbolic. Nevertheless, the use of the second conserved quantity shows that the trajectories are intersections of the energy ellipsoids $T = \text{constant}$ with the sphere $J = 1$, which implies that these equilibria are nonlinear centers and are stable (but not asymptotically stable).

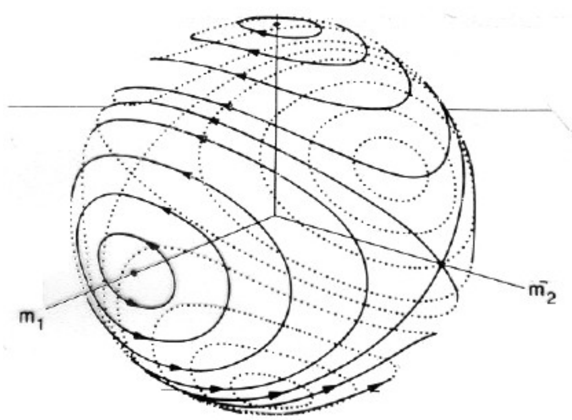

Figure 1: Phase portrait for rigid body rotation (from Bender and Orzag).