

Dynamical systems and ODEs

The subject of dynamical systems concerns the evolution of systems in time. In continuous time, the systems may be modeled by ordinary differential equations (ODEs), partial differential equations (PDEs), or other types of equations (e.g., integro-differential or delay equations); in discrete time, they may be modeled by difference equations or iterated maps.

Linear evolution equations have an extensive theory based on the superposition principle that every linear combination of solutions is also a solution. Using this principle, one can often obtain general solutions as linear combinations of sufficiently many special solutions, which one may be able to find more-or-less explicitly.

Nonlinear evolution equations are not explicitly solvable, except in rare but important special cases, and the solutions of even simple-looking nonlinear equations can exhibit complicated behavior such as chaos. The focus of dynamical systems is to understand the qualitative behavior of the solutions. Typical questions include: What are the equilibrium or time-periodic solutions? Are these solutions stable? What is the long-time asymptotic behavior of general solutions? Do solutions behave chaotically? What kinds of statistical regularities do solutions possess? How does the qualitative dynamics change as any parameters on which the system depends vary? Do equilibria lose stability, do time-periodic solutions appear, or does chaotic behavior arise?

These notes are concerned with low-dimensional dynamical systems, whose state is described by a few variables. The evolution of these system may be described by ODEs or maps on a low-dimensional state-space. Dynamical systems with high or infinite-dimensional state spaces, such as PDEs, can show many types of behavior that do not arise in low-dimensional systems (e.g., solutions with multiscale spatial structures such as turbulence).

We will begin by discussing some general properties of initial value problems (IVPs) for ODEs as well as their basic underlying mathematical theory.

1.1. First-order systems of ODEs

Does the Flap of a Butterfly's
Wings in Brazil Set off a Tornado
in Texas?

Edward Lorenz, 1972

An autonomous system of first-order ODEs has the form

$$(1.1) \quad x_t = f(x)$$

where $x(t) \in \mathbb{R}^d$ is a vector of dependent variables, $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector field, and x_t is the time-derivative, which we also write as dx/dt or \dot{x} . We may regard

(1.1) as describing the evolution in continuous time t of a dynamical system with finite-dimensional state $x(t)$ of dimension d .

In component form, we write $x = (x_1, \dots, x_d)$,

$$f(x) = (f_1(x_1, \dots, x_d), \dots, f_n(x_1, \dots, x_d)),$$

and the system (1.1) is

$$\begin{aligned} x_{1t} &= f_1(x_1, \dots, x_d), \\ x_{2t} &= f_2(x_1, \dots, x_d), \\ &\vdots \\ x_{dt} &= f_d(x_1, \dots, x_d). \end{aligned}$$

Usually, we do not use notation such as \vec{x} to distinguish vectors explicitly from scalars, and we write vectors as row or column vectors as convenient.

Autonomous ODEs arise as models of systems whose laws do not change in time. They are invariant under translations in time: if $x(t)$ is a solution, then so is $x(t + t_0)$ for any constant t_0 .

EXAMPLE 1.1. The Lorenz system for $(x, y, z) \in \mathbb{R}^3$ is

$$(1.2) \quad \begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - bz. \end{aligned}$$

The system depends on three positive parameters σ , r , b ; a commonly studied case is $\sigma = 10$, $r = 28$, and $b = 4/3$. Lorenz (1963) obtained (1.2) as a truncated model of thermal convection in a fluid layer, where σ has the interpretation of a Prandtl number (the ratio of kinematic viscosity and thermal diffusivity), r corresponds to a Rayleigh number, which is a dimensionless parameter proportional to the temperature difference across the fluid layer and the gravitational acceleration acting on the fluid, and b is a ratio of the height and width of the fluid layer. (See Section 1.8.)

Lorenz discovered that solutions of (1.2) behave chaotically, showing that even low-dimensional nonlinear dynamical systems can behave in complex ways. Solutions of chaotic systems are sensitive to small changes in the initial conditions, and Lorenz used this model to discuss the unpredictability of weather (the “butterfly effect”).

If $\bar{x} \in \mathbb{R}^d$ is a zero of f , meaning that

$$(1.3) \quad f(\bar{x}) = 0,$$

then (1.1) has the constant solution $x(t) = \bar{x}$. We call \bar{x} an *equilibrium solution*, or *steady state solution*, or *fixed point* of (1.1). An equilibrium may be stable or unstable, depending on whether small perturbations of the equilibrium decay — or, at least, remain bounded — or grow. (See Definition 1.22 below for a precise definition.) The determination of the stability of equilibria will be an important topic in the following.

Other types of ODEs can be put in the form (1.1). This rewriting does not simplify their analysis, and may obscure the specific structure of the ODEs, but it shows that (1.1) is rather a general form.

EXAMPLE 1.2. A non-autonomous system for $x(t) \in \mathbb{R}^d$ has the form

$$(1.4) \quad x_t = f(x, t)$$

where $f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$. A nonautonomous ODE describes systems governed by laws that vary in time, e.g., due to external influences. Equation (1.4) can be written as an autonomous ('suspended') system for $y = (x, s) \in \mathbb{R}^{n+1}$ with $s = t$ as

$$x_t = f(x, s), \quad s_t = 1.$$

Note that this increases the dimension of the system by one. Moreover, even if the original system has an equilibrium solution $x(t) = \bar{x}$ such that $f(\bar{x}, t) = 0$, the suspended system has no equilibrium solutions for y .

Higher-order ODEs can be written as first order systems by the introduction of derivatives as new dependent variables.

EXAMPLE 1.3. A second-order system for $x(t) \in \mathbb{R}^d$ of the form

$$(1.5) \quad x_{tt} = f(x, x_t)$$

can be written as a first-order system for $z = (x, y) \in \mathbb{R}^{2d}$ with $y = x_t$ as

$$x_t = y, \quad y_t = f(x, y).$$

Note that this doubles the dimension of the system.

EXAMPLE 1.4. In Newtonian mechanics, the position $x(t) \in \mathbb{R}^d$ of a particle of mass m moving in d space dimensions in a spatially-dependent force-field $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies Newton's second law,

$$mx_{tt} = F(x).$$

If $p = mx_t$ is the momentum of the particle, then (x, p) satisfies the first-order system

$$(1.6) \quad x_t = \frac{1}{m}p, \quad p_t = F(x).$$

A conservative force-field is derived from a scalar-valued potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$F = -\frac{\partial V}{\partial x}, \quad (F_1, \dots, F_d) = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_d} \right),$$

where we use $\partial/\partial x$, $\partial/\partial p$ to denote the derivatives, or gradients, with respect to x , p respectively. For example, the gravitational potential of a planet moving around the sun (which we assume to be fixed at the origin) is given by

$$(1.7) \quad V(x) = -\frac{GmM}{|x|},$$

where $G \approx 6.673 \times 10^{-11} \text{ Nm/kg}^2$ is the gravitational constant, m is the mass of the planet, and M is the mass of the sun. In that case,

$$F(x) = -\frac{GmM}{|x|^2} \frac{x}{|x|},$$

meaning that the gravitational force on the planet is directed towards the sun with strength proportional to $|x|^{-2}$. (Newton's inverse-square law of gravity.)

For a conservative force field, (1.6) becomes the Hamiltonian system

$$(1.8) \quad x_t = \frac{\partial H}{\partial p}, \quad p_t = -\frac{\partial H}{\partial x}$$

where the Hamiltonian

$$(1.9) \quad H(x, p) = \frac{1}{2m}p^2 + V(x)$$

is the total energy (kinetic + potential) of the particle. The Hamiltonian is a conserved quantity of (1.8), since by the chain rule

$$\begin{aligned} \frac{d}{dt}H(x(t), p(t)) &= \frac{\partial H}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial H}{\partial p} \cdot \frac{dp}{dt} \\ &= -\frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial x} \\ &= 0. \end{aligned}$$

Thus, solutions (x, p) of (1.8) lie on the level surfaces $H(x, p) = \text{constant}$.

1.2. Existence and uniqueness theorem for IVPs

An initial value problem (IVP) for (1.1) consists of solving the ODE subject to an initial condition (IC) for x :

$$(1.10) \quad \begin{aligned} x_t &= f(x), \\ x(0) &= x_0. \end{aligned}$$

Here, $x_0 \in \mathbb{R}^d$ is a given constant vector. For an autonomous system, there is no loss of generality in imposing the initial condition at $t = 0$, rather than some other time $t = t_0$.

For a first-order system, we impose initial data for x . For a second-order system, such as (1.5), we impose initial data for x and x_t , and analogously for higher-order systems. The ODE in (1.10) determines $x_t(0)$ from x_0 , and we can obtain all higher order derivatives $x^{(n)}(0)$ by differentiating the ODE with respect to t and evaluating the result at $t = 0$. Thus, it is reasonable to expect that (1.10) determines a unique solution, and this is indeed true provided that $f(x)$ satisfies a mild smoothness condition, called Lipschitz continuity, which is nearly always met in applications. Before stating the existence-uniqueness theorem, we explain what Lipschitz continuity means.

We denote by

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

the Euclidean norm of a vector $x \in \mathbb{R}^d$.

DEFINITION 1.5. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is *locally Lipschitz continuous* on \mathbb{R}^d , or Lipschitz continuous for short, if for every $R > 0$ there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in \mathbb{R}^d \text{ such that } |x|, |y| \leq R.$$

The function f is *globally Lipschitz continuous* on \mathbb{R}^d if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in \mathbb{R}^d$$

We refer to the constant M in this definition as a Lipschitz constant for f . Every continuously differentiable function is locally Lipschitz continuous.

DEFINITION 1.6. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, where

$$f(x) = (f_1(x_1, x_2, \dots, x_d), f_2(x_1, x_2, \dots, x_d), \dots, f_d(x_1, x_2, \dots, x_d)),$$

is continuously differentiable on \mathbb{R}^d if all its partial derivatives

$$\frac{\partial f_i}{\partial x_j}, \quad 1 \leq i, j \leq d$$

exist and are continuous functions.

PROPOSITION 1.7. A continuously differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz continuous on \mathbb{R}^n .

PROOF. From the fundamental theorem of calculus

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \frac{d}{ds} f(y + s(x - y)) ds \\ &= \int_0^1 Df(y + s(x - y)) (x - y) ds. \end{aligned}$$

Here Df is the derivative of f , whose matrix is the Jacobian matrix of f with components $\partial f_i / \partial x_j$. Hence

$$\begin{aligned} |f(x) - f(y)| &\leq \int_0^1 |Df(y + s(x - y)) (x - y)| ds \\ &\leq \left(\int_0^1 \|Df(y + s(x - y))\| ds \right) |x - y| \\ &\leq M|x - y| \end{aligned}$$

where $\|Df\|$ denotes the Euclidean matrix norm of Df and

$$M = \max_{0 \leq s \leq 1} \|Df(y + s(x - y))\|,$$

which is finite since Df is continuous. For scalar-valued functions, this result also follows from the mean value theorem. \square

EXAMPLE 1.8. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is locally Lipschitz continuous on \mathbb{R} , since it is continuously differentiable. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = |x|$ is Lipschitz continuous, although it is not differentiable at $x = 0$. The function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = |x|^{1/2}$ is not Lipschitz continuous at $x = 0$, although it is continuous.

The following result, due to Picard and Lindelöf, is the fundamental local existence and uniqueness theorem for IVPs for ODEs. It is a local existence theorem because it only asserts the existence of a solution for sufficiently small times, not necessarily for all times.

THEOREM 1.9 (Existence-uniqueness). If $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz continuous, then there exists a unique solution $x : I \rightarrow \mathbb{R}^d$ of (1.10) defined on some time-interval $I \subset \mathbb{R}$ containing $t = 0$.

In practice, to apply this theorem to (1.10), we usually just have to check that the right-hand side $f(x)$ is a continuously differentiable function of the dependent variables x . We also remark that this theorem is a local result, and it is sufficient for f to be Lipschitz continuous or continuously differentiable in some neighborhood of the initial data x_0 .

We will not prove Theorem 1.9 here, but we explain the main idea of the proof. Since it is impossible, in general, to find an explicit solution of a nonlinear IVP such as (1.10), we have to construct the solution by some kind of approximation procedure. Using the method of Picard iteration, we rewrite (1.10) as an equivalent integral equation

$$(1.11) \quad x(t) = x_0 + \int_0^t f(x(s)) \, ds.$$

This integral equation formulation includes both the initial condition and the ODE. We then define a sequence $x_n(t)$ of functions by iteration, starting from the constant initial data x_0 :

$$(1.12) \quad x_{n+1}(t) = x_0 + \int_0^t f(x_n(s)) \, ds, \quad n = 1, 2, 3, \dots$$

Using the Lipschitz continuity of f , one can show that this sequence converges uniformly on a sufficiently small time interval I to a unique function $x(t)$, and taking the limit of (1.12) as $n \rightarrow \infty$, we find that $x(t)$ satisfies (1.11), so it is the solution of (1.10).

Two simple scalar examples illustrate Theorem 1.9. The first example shows that solutions of nonlinear IVPs with smooth vector fields may develop singularities in finite time. We refer to this phenomenon informally as ‘blow-up,’ and it is a fundamental difficulty in the analysis of nonlinear evolution equations.

EXAMPLE 1.10. Consider the IVP

$$x_t = x^2, \quad x(0) = x_0.$$

For $x_0 \neq 0$, we find by separating variables that the solution is

$$(1.13) \quad x(t) = \frac{1}{1/x_0 - t}.$$

If $x_0 > 0$, the solution exists only for $-\infty < t < t_*$ where $t_* = 1/x_0$, and $x(t) \rightarrow \infty$ as $t \uparrow t_*$. Note that the larger the initial data x_0 the smaller the ‘blow-up’ time t_* . If $x_0 < 0$, then $t_* < 0$ and the solution exists for $t_* < t < \infty$. Only if $x_0 = 0$ does the solution $x(t) = 0$ exist for all times $t \in \mathbb{R}$.

One might consider using (1.13) past the blow-up time, but continuing a solution through infinity does not usually make sense in evolution problems. In applications, the appearance of a singularity typically signifies that the assumptions of the mathematical model have broken down in some way.

The second example shows that solutions of (1.10) need not be unique if f is not Lipschitz continuous.

EXAMPLE 1.11. Consider the IVP

$$(1.14) \quad x_t = |x|^{1/2}, \quad x(0) = 0.$$

The right-hand side of the ODE, $f(x) = |x|^{1/2}$, is not differentiable or Lipschitz continuous at the initial value $x = 0$. One solution of (1.14) is $x(t) = 0$, but this is not the only solution. Separating variables in the ODE, we get the solution

$$x(t) = \frac{1}{4}(t - t_0)^2.$$

Thus, for any $t_0 \geq 0$, the function

$$x(t) = \begin{cases} 0 & \text{if } t \leq t_0 \\ (1/4)(t - t_0)^2 & \text{if } t > t_0 \end{cases}$$

is also a continuously-differentiable solution of the IVP (1.14). The parabolic solution can ‘take off’ spontaneously with zero derivative from the zero solution at any nonnegative time t_0 . In applications, a lack of uniqueness typically means that something is missing from the formulation of the mathematical model.

If $f(x)$ is only assumed to be a continuous function of x , then solutions of (1.10) always exist (this is the Peano existence theorem) although they may fail to be unique, as shown by Example 1.11. In future, we will assume that f is a smooth function; typically, f will be C^∞ , meaning that it has continuous derivatives of all orders. In that case, the issue of non-uniqueness of solutions of IVPs does not arise.

1.3. Extension of solutions

Even for arbitrarily smooth functions f , the solution of the nonlinear IVP (1.10) may fail to exist for all times if $f(x)$ grows faster than a linear function of x , as shown in Example 1.10. According to the following theorem, the only way in which global existence can fail for a smooth system of ODEs on \mathbb{R}^d is if the solution ‘escapes’ to infinity.

THEOREM 1.12 (Extension). *If $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz continuous on \mathbb{R}^d , then the solution $x : I \rightarrow \mathbb{R}^d$ of the initial value problem (1.10) exists on a maximal time-interval*

$$I = (T_-, T_+) \subset \mathbb{R}$$

where $-\infty \leq T_- < 0$ and $0 < T_+ \leq \infty$ depend, in general, on the initial data x_0 . If $T_+ < \infty$, then $|x(t)| \rightarrow \infty$ as $t \uparrow T_+$, and if $T_- > -\infty$, then $|x(t)| \rightarrow \infty$ as $t \downarrow T_-$.

This theorem implies that we can continue a solution of the ODE so long as it remains bounded.

EXAMPLE 1.13. Consider the function defined for $t \neq 0$ by

$$x(t) = \sin\left(\frac{1}{t}\right).$$

This function cannot be extended to a differentiable, or even continuous, function at $t = 0$ even though it is bounded, so $x(t)$ is not a solution of any ODE $x_t = f(x)$ with a smooth right-hand side f on any time interval $|t| < T$. This is impossible because the ODE implies that the derivative x_t remains bounded so long as the solution x remains bounded. On the other hand, an ODE may have a solution like $x(t) = 1/t$, since the derivative x_t only becomes large when x itself becomes large.

EXAMPLE 1.14. Consider the Hamiltonian system (1.8) with Hamiltonian (1.9). If $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, then $H(x, p) = \text{constant}$ implies that (x, p) remains bounded, so solutions exist for all $t \in \mathbb{R}$.

EXAMPLE 1.15. Consider the motion of a mass in the gravitational potential (1.7). The position $x(t) \in \mathbb{R}^3$ of the mass satisfies

$$x_{tt} = f(x), \quad f(x) = -\frac{GM}{|x|^2} \frac{x}{|x|}.$$

It is not possible to extend a solution smoothly if $x(t) \rightarrow 0$ in finite time, meaning that the mass collides with the gravitational point-source at the origin. This loss of global existence occurs because the non-smooth vector field $f(x)$ has a singularity at $x = 0$, not because the solution escapes to infinity.

EXAMPLE 1.16. Theorem 1.9 implies that the Lorenz system (1.1) with arbitrary initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0$$

has a unique solution defined on some time interval containing 0, since the right hand side is a smooth (in fact, quadratic) function of (x, y, z) . The theorem does not imply, however, that the solution exists for all t .

Nevertheless, we claim that when the parameters (σ, r, b) are positive the solution does exist for all $t \geq 0$. From Theorem 1.12, this conclusion follows if we can show that the solution remains bounded, and to do this we introduce a suitable Lyapunov function. A convenient choice is

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2.$$

Using the chain rule, we find that if (x, y, z) satisfies (1.2), then

$$\begin{aligned} \frac{d}{dt}V(x, y, z) &= 2rxx_t + 2\sigma yy_t + 2\sigma(z - 2r)z_t \\ &= 2r\sigma x(y - x) + 2\sigma y(rx - y - xz) + 2\sigma(z - 2r)(xy - bz) \\ &= -2\sigma [rx^2 + y^2 + b(z - r)^2] + 2b\sigma r^2. \end{aligned}$$

Hence, if $W(x, y, z) > br^2$, where

$$W(x, y, z) = rx^2 + y^2 + b(z - r)^2,$$

then $V(x, y, z)$ is decreasing in time. This means that if C is sufficiently large that the ellipsoid $V(x, y, z) < C$ contains the ellipsoid $W(x, y, z) \leq br^2$, then solutions cannot escape from the region $V(x, y, z) < C$ forward in time, since they move ‘inwards’ across the boundary $V(x, y, z) = C$. Therefore, the solution remains bounded and exists for all $t \geq 0$.

Note that this argument does not preclude the possibility that solutions of (1.2) blow up backwards in time. The Lorenz system models a forced, dissipative system and its dynamics are not time-reversible. (This contrasts with the dynamics of conservative, Hamiltonian systems, which are time-reversible.)

The phenomenon of blow-up also occurs for solutions of nonlinear PDEs, although the situation there is more complicated because spatial derivatives of the solution may become unbounded, rather than the solution itself.

EXAMPLE 1.17. The following PDE for $u(x, t)$

$$u_t + uu_x = 0$$

is called the inviscid Burgers equation. The method of characteristics shows that u remains bounded but u_x typically approaches $-\infty$ in finite time. One can continue smooth solutions past the blow-up time by introducing suitable discontinuities, called shocks, in the solutions [15].

EXAMPLE 1.18. The velocity $\vec{u}(\vec{x}, t)$ and pressure $p(\vec{x}, t)$ of a viscous incompressible fluid satisfy the incompressible Navier-Stokes equations

$$\begin{aligned}\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho_0} \nabla p &= \nu \Delta \vec{u}, \\ \nabla \cdot \vec{u} &= 0,\end{aligned}$$

where the constants ρ_0 and ν are the density and kinematic viscosity of the fluid, respectively. In three-space dimensions, it is unknown whether or not the derivatives $\nabla \vec{u}$ can develop a singularity in finite time. The resolution of this question is one of the Clay Mathematics Institute Millennium Problems [2]. If the derivatives of \vec{u} do blow up, then solutions of the initial value problem (IVP) for the Navier-Stokes equations may not be unique, in which case the IVP is not well-posed and the Navier-Stokes equations would not provide a satisfactory mathematical model for fluid flow in such cases.

1.4. Linear systems of ODEs

An IVP for a (homogeneous, autonomous, first-order) linear system of ODEs for $x(t) \in \mathbb{R}^d$ has the form

$$(1.15) \quad \begin{aligned}x_t &= Ax, \\ x(0) &= x_0\end{aligned}$$

where A is a $d \times d$ matrix and $x_0 \in \mathbb{R}^d$. This system corresponds to (1.10) with $f(x) = Ax$. Linear systems are much simpler to study than nonlinear systems, and perhaps the first question to ask of any equation is whether it is linear or nonlinear.

The linear IVP (1.15) has a unique global solution, which is given explicitly by

$$x(t) = e^{tA} x_0, \quad -\infty < t < \infty$$

where

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \dots + \frac{1}{n!}t^nA^n + \dots$$

is the matrix exponential.

If A is nonsingular, then (1.15) has a unique equilibrium solution $x = 0$. This equilibrium is stable if all eigenvalues of A have negative real parts and unstable if some eigenvalue of A has positive real part. If A is singular, then there is a ν -dimensional subspace of equilibria where ν is the nullity of A .

Linear systems are important in their own right, but they also arise as approximations of nonlinear systems. Suppose that \bar{x} is an equilibrium solution of (1.1), satisfying (1.3). Then writing

$$x(t) = \bar{x} + y(t)$$

and Taylor expanding $f(x)$ about \bar{x} , we get

$$f(\bar{x} + y) = Ay + \dots$$

where A is the derivative of f evaluated at \bar{x} , with matrix (a_{ij}) :

$$A = Df(\bar{x}), \quad a_{ij} = \frac{\partial f_i}{\partial x_j}(\bar{x}).$$

The linearized approximation of (1.1) at the equilibrium \bar{x} is then

$$y_t = Ay.$$

An important question is if this linearized system provides a good local approximation of the nonlinear system for solutions that are near equilibrium. This is the case under the following condition.

DEFINITION 1.19. An equilibrium \bar{x} of (1.1) is hyperbolic if $Df(\bar{x})$ has no eigenvalues with zero real part.

Thus, for a hyperbolic equilibrium, all solutions of the linearized system grow or decay exponentially in time. According to the Hartman-Grobman theorem, if \bar{x} is hyperbolic, then the flows of the linearized and nonlinear system are (topologically) equivalent near the equilibrium. In particular, the stability of the nonlinear equilibrium is the same as the stability of the equilibrium of the linearized system. One has to be careful, however, in drawing conclusions about the behavior of the nonlinear system from the linearized system if $Df(\bar{x})$ has eigenvalues with zero real part. In that case the nonlinear terms may cause the growth or decay of perturbations from equilibrium, and the behavior of solutions of the nonlinear system near the equilibrium may differ qualitatively from that of the linearized system.

Non-hyperbolic equilibria are not typical for specific systems, since one does not expect the eigenvalues of a given matrix to have a real part that is exactly equal to zero. Nevertheless, non-hyperbolic equilibria arise in an essential way in bifurcation theory when an eigenvalue of a system that depends on some parameter has real part that passes through zero.

1.5. Phase space

it may happen that small
differences in the initial conditions
produce very great ones in the final
phenomena

Henri Poincaré, 1908

Very few nonlinear systems of ODEs are explicitly solvable. Therefore, rather than looking for individual analytical solutions, we try to understand the qualitative behavior of their solutions. This global, geometrical approach was introduced by Poincaré (1880).

We may represent solutions of (1.10) by solution curves, trajectories, or orbits, $x(t) \in \mathbb{R}^d$ in phase, or state, space.¹ These trajectories are integral curves of the vector field f , meaning that they are tangent to f at every point. The existence-uniqueness theorem implies if the vector field $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth, then a unique trajectory passes through each point of phase space \mathbb{R}^d and that trajectories cannot cross. We may visualize f as the steady velocity field of a fluid that occupies phase space and the trajectories as the paths of fluid particles.

Let $x(t; x_0)$ denote the solution of (1.10), defined on its maximal time-interval of existence $T_-(x_0) < t < T_+(x_0)$. The existence-uniqueness theorem implies that for each $t \in \mathbb{R}$ we can define a flow, or solution, map $\Phi_t : U_t \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$\Phi_t(x_0) = x(t; x_0), \quad U_t = \{x_0 \in \mathbb{R}^d : T_-(x_0) < t < T_+(x_0)\}.$$

¹Sometimes the term ‘phase space’ is used to refer specifically to the state space of a Hamiltonian system; we will use the term to refer to the state space of any dynamical system.

That is, Φ_t maps the initial data x_0 to the solution at time t .² In the fluid analogy, Φ_t may be interpreted as the map that takes a particle from its initial location at time 0 to its location at time t . Note that $\Phi_t(x_0)$ is not defined for all $t \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ unless all solutions exist globally.

The next result implies that the solution of an IVP depends continuously on the initial data, and that the flow map of a smooth vector field is smooth. Here, ‘smooth’ means, for example, C^1 or C^∞ .

THEOREM 1.20 (Continuous dependence on initial data). *If the vector field f in (1.10) is locally Lipschitz continuous, then the corresponding flow map $\Phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz continuous. Moreover, the existence times T_+ (respectively, T_-) are lower (respectively, upper) semi-continuous function of x_0 . If the vector field f in (1.10) is smooth, then the corresponding flow map $\Phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth.*

Here, the lower-semicontinuity of T_+ means that

$$T_+(x_0) \leq \liminf_{x \rightarrow x_0} T_+(x),$$

so that solutions with initial data near x_0 exist for essentially as long, or perhaps longer, than the solution with initial data x_0 .

We will not prove Theorem 1.20 here, but we indicate why the flow map of a Lipschitz continuous vector field is Lipschitz continuous. If M is the Lipschitz constant of f and $x(t)$, $y(t)$ are two solutions of (1.10), then

$$\frac{d}{dt}|x - y| \leq M|x - y|.$$

It follows from Gronwall’s inequality that if $x(0) = x_0$, $y(0) = y_0$, then

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{Mt}.$$

Thus, solutions remain close over a finite time-interval if their initial data are sufficiently close. After long enough times, however, two solutions may diverge by an arbitrarily large amount however close their initial data.

EXAMPLE 1.21. Consider the scalar, linear ODE $\dot{x}_t = x$. The solutions $x(t)$, $y(t)$ with initial data $x(0) = x_0$, $y(0) = y_0$ are given by

$$x(t) = x_0e^t, \quad y(t) = y_0e^t.$$

Suppose that $[0, T]$ is any given time interval, where $T > 0$. If $|x_0 - y_0| \leq \epsilon e^{-T}$, then the solutions satisfy $|x(t) - y(t)| \leq \epsilon$ for all $0 \leq t \leq T$, so the solutions remain close on $[0, T]$, but $|x(t) - y(t)| \rightarrow \infty$ as $t \rightarrow \infty$ whenever $x_0 \neq y_0$.

The local exponential divergence (or contraction) of trajectories of a nonlinear system may be different in different directions, and is measured by the Lyapunov exponents of the system. The largest such exponent is called *the* Lyapunov exponent of the system. Chaotic behavior occurs in systems with a strictly positive Lyapunov exponent whose trajectories remain bounded; it is associated with the local exponential divergence of trajectories (essentially a linear phenomenon) followed by a global folding (typically as a result of nonlinearity).

²Here, it is convenient to use a t -subscript in Φ_t and U_t to indicate dependence on the time t ; the subscript does not denote derivative with respect to t .

The flow map Φ_t of an autonomous system has the group property that

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

where \circ denotes the composition of maps, i.e., solving the ODE for time $t + s$ is equivalent to solving it for time s , then for time t . Also, $\Phi_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the identity map $\Phi_0(x_0) = x_0$.

We remark that the solution map of a non-autonomous IVP,

$$x_t = f(x, t), \quad x(t_0) = x_0$$

with solution $x(t; x_0, t_0)$, is defined by

$$\Phi_{t, t_0}(x_0) = x(t; x_0, t_0).$$

The map depends on both the initial and final time, not just their difference, and satisfies (where defined)

$$\Phi_{t, s} \circ \Phi_{s, r} = \Phi_{t, r}.$$

If \bar{x} is an equilibrium solution of (1.10), with $f(\bar{x}) = 0$, then

$$\Phi_t(\bar{x}) = \bar{x},$$

which explains why equilibria are also referred to as fixed points (of the flow map). We may state a precise definition of the stability of an equilibrium in terms of the flow map. There are many different, and not entirely equivalent definitions, of stability; we give only the two simplest and most commonly used ones.

DEFINITION 1.22. An equilibrium \bar{x} of (1.10) is Lyapunov stable (or stable, for short) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - \bar{x}| < \delta$ then

$$|\Phi_t(x) - \bar{x}| < \epsilon \quad \text{for all } t \geq 0.$$

The equilibrium is asymptotically stable if it is Lyapunov stable and there exists $\eta > 0$ such that if $|x - \bar{x}| < \eta$ then

$$\Phi_t(x) \rightarrow \bar{x} \quad \text{as } t \rightarrow \infty.$$

Thus, stability means that solutions remain arbitrarily close to the equilibrium for all $t \geq 0$ if they start sufficiently close to the equilibrium, while asymptotic stability means that, in addition, nearby solutions approach the equilibrium as $t \rightarrow \infty$. Lyapunov stability does not imply asymptotic stability since, for example, nearby solutions might oscillate about an equilibrium without decaying toward it. Also, it is not sufficient for asymptotic stability that all solutions with nearby initial data approach the equilibrium, because they could make large excursions before approaching the equilibrium, which would violate Lyapunov stability.

One way to organize the study of dynamical systems is by the dimension of their phase space (following the Trolls of Discworld: one, two, three, many, and lots). In one or two dimensions, the non-intersection of trajectories imposes strong topological constraints on their possible behavior. In one dimension, solutions can only increase or decrease monotonically to an equilibrium or to infinity; in two dimensions, oscillatory, periodic behavior can occur, but solutions cannot cross from the inside to the outside (or visa-versa) of a closed, periodic orbit. In three or more dimensions complex behavior is possible; for example, a trajectory can wander around a bounded region of phase space without self-intersections in an irregular and chaotic fashion.

For the most part, we will consider dynamical systems with low-dimensional phase spaces (of dimension $d \leq 3$). The analysis of high-dimensional dynamical systems is usually very difficult, and may require (more or less well-founded) probabilistic assumptions, or continuum approximations, or some other type of approach.

EXAMPLE 1.23. Consider a gas composed of N classical particles of mass m moving in three space dimensions with an interaction potential $V : \mathbb{R}^3 \rightarrow \mathbb{R}$. We denote the positions of the particles by $x = (x_1, x_2, \dots, x_N)$ and the momenta by $p = (p_1, p_2, \dots, p_N)$, where $x_i, p_i \in \mathbb{R}^3$. The Hamiltonian for this system is

$$H(x, p) = \frac{1}{2m} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V(x_i - x_j),$$

and Hamilton's equations are

$$\frac{dx_i}{dt} = \frac{1}{m} p_i, \quad \frac{dp_i}{dt} = - \sum_{j \neq i} \frac{\partial V}{\partial x} (x_i - x_j).$$

The phase space of this system has dimension $6N$. For a mole of gas, we have $N = N_A$ where $N_A \approx 6.02 \times 10^{23}$ is Avogadro's number, and this dimension is extremely large. In kinetic theory, one considers equations for probability distributions of the particle locations and velocities, such as the Vlasov or Boltzmann equations. One can also approximate some solutions by partial differential fluid equations, such as the Euler or Navier-Stokes equations, for suitable averages.

We will mostly consider systems whose phase space is \mathbb{R}^d . More generally, the phase space of a dynamical system may be a manifold. We will not give a precise definition of a manifold here; roughly speaking, a d -dimensional manifold is a space that 'looks' locally like \mathbb{R}^d , with a d -dimensional local coordinate system about each point, but which may have a different global, topological structure. The d -dimensional sphere

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$$

is a typical example. Phase spaces that are manifolds arise, for example, if some of the state variables represent angles.

EXAMPLE 1.24. The motion of an undamped pendulum of length ℓ in a gravitational field with acceleration g satisfies the pendulum equation

$$\theta_{tt} + \frac{g}{\ell} \sin \theta = 0$$

where $\theta \in \mathbb{T}$ is the angle of the pendulum to the vertical, measured in radians. Here, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denotes the circle; angles that differ by an integer multiple of 2π are equivalent. Writing the pendulum equation as a first-order system for (θ, v) where $v = \theta_t \in \mathbb{R}$ is the angular velocity, we get

$$\theta_t = v, \quad v_t = -\frac{g}{\ell} \sin \theta.$$

The phase space of this system is the cylinder $\mathbb{T} \times \mathbb{R}$. This phase space may be 'unrolled' into \mathbb{R}^2 with points on the θ -axis identified modulo 2π , but it is often conceptually clearer to keep the actual cylindrical structure and θ -periodicity in mind.

EXAMPLE 1.25. The phase space of a rotating rigid body, such as a tumbling satellite, may be identified with the group $SO(3)$ of rotations about its center of mass from some fixed reference configuration. The three Euler angles of a rotation give one possible local coordinate system on the phase space.

Solutions of an ODE with a smooth vector field on a compact phase space without boundaries, such as \mathbb{S}^d , exist globally in time since they cannot escape to infinity (or hit a boundary).

1.6. Bifurcation theory

Most applications lead to equations which depend on parameters that characterize properties of the system being modeled. We write an IVP for a first-order system of ODEs for $x(t) \in \mathbb{R}^d$ depending on a vector of parameters $\mu \in \mathbb{R}^m$ as

$$(1.16) \quad \begin{aligned} x_t &= f(x; \mu), \\ x(0) &= x_0 \end{aligned}$$

where $f : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d$.

In applications, it is important to determine a minimal set of dimensionless parameters on which the problem depends and to know what parameter regimes are relevant, e.g., whether some dimensionless parameters are very large or small.

EXAMPLE 1.26. The Lorentz system (1.2) for $(x, y, z) \in \mathbb{R}^3$ depends on three parameters $(\sigma, r, b) \in \mathbb{R}^3$. We typically think of fixing (σ, b) and increasing r , which in the original convection problem corresponds to fixing the fluid properties and the dimensions of the fluid layer and increasing the temperature difference across it.

If the vector field in (1.16) depends smoothly (e.g., C^1 or C^∞) on the parameter μ , then so does the flow map. Explicitly, if $x(t; x_0; \mu)$ denotes the solution of (1.16), then we define the flow map Φ_t by

$$\Phi_t(x_0; \mu) = x(t; x_0; \mu).$$

THEOREM 1.27 (Continuous dependence on parameters). *If the vector field $f : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ in (1.16) is smooth, then the corresponding flow map Φ_t is smooth.*

Bifurcation theory is concerned with changes in the qualitative behavior of the solutions of (1.16) as the parameter μ is varied. It may be difficult to carry out a full bifurcation analysis of a nonlinear dynamical system, especially when it depends on many parameters.

The simplest type of bifurcation is the bifurcation of equilibria. The equilibrium solutions of (1.16) satisfy

$$f(\bar{x}; \mu) = 0,$$

so an analysis of equilibrium bifurcations corresponds to understanding how the solutions $\bar{x}(\mu) \in \mathbb{R}^d$ of this $d \times d$ system of nonlinear, algebraic equations depend upon the parameter μ . We refer to a smooth solution $\bar{x} : I \rightarrow \mathbb{R}^d$ defined on a maximal domain $I \subset \mathbb{R}^m$ as a solution branch or a branch of equilibria.

There is a closely related dynamical aspect concerning how the stability of the equilibria change as the parameter μ varies. If $\bar{x}(\mu)$ is a branch of equilibrium solutions, then the linearization of the system about \bar{x} is

$$x_t = A(\mu)x, \quad A(\mu) = D_x f(\bar{x}(\mu); \mu).$$

Equilibria lose stability if some eigenvalue $\lambda(\mu)$ of A crosses from the left-half of the complex plane into the right-half plane as μ varies. By the implicit function theorem, equilibrium bifurcations are necessarily associated with a real eigenvalue passing through zero, so that A must be singular at the bifurcation point.

Equilibrium bifurcations are not the only kind of bifurcation, and the dynamic behavior of a system may change without a change in the equilibrium solutions. For example, time-periodic solutions may appear or disappear in a Hopf bifurcation, which occurs where a complex-conjugate pair of complex eigenvalues of A crosses into the right-half plane, or there may be global changes in the geometry of the trajectories in phase space, as in a homoclinic bifurcation.

1.7. Discrete dynamical systems

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessarily possess simple dynamical properties

Robert May, 1976

A (first-order, autonomous) discrete dynamical system for $x_n \in \mathbb{R}^d$ has the form

$$(1.17) \quad x_{n+1} = f(x_n)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $n \in \mathbb{Z}$ is a discrete time variable.

An orbit, or trajectory of (1.17) consist of a sequence of points $\{x_n\}$ that is obtained by iterating the map f . (These orbits are sequences of points, not curves like the orbits of a continuous dynamical system.) If $f^n = f \circ f \circ \dots \circ f$ denotes the n -fold composition of f , then

$$x_n = f^n(x_0).$$

If f is invertible, then these orbits exists forward and backward in time (for $n \in \mathbb{Z}$), while if f is not invertible, then in general they exist only forward in time (for $n \in \mathbb{N}_0$). An equilibrium solution \bar{x} of (1.17) is a fixed point of f that satisfies

$$f(\bar{x}) = \bar{x},$$

and in that case $x_n = \bar{x}$ for all n .

A linear discrete dynamical system has the form

$$(1.18) \quad x_{n+1} = Bx_n,$$

where B is a linear transformation on \mathbb{R}^d . The solution is

$$x_n = B^n x_0.$$

The linear system (1.18) has the unique fixed point $\bar{x} = 0$ if $I - B$ is a nonsingular linear map. This fixed point is asymptotically stable if all eigenvalues $\lambda \in \mathbb{C}$ of B lie in the unit disc, meaning that $|\lambda| < 1$. It is unstable if B has some eigenvalue with $|\lambda| > 1$ in the exterior of the unit disc.

The linearization of (1.17) about a fixed point \bar{x} is

$$x_{n+1} = Bx_n, \quad B = Df(\bar{x}).$$

Analogously to the case of continuous systems, we can determine the stability of the fixed point from the stability of the linearized system under a suitable hyperbolicity assumption.

DEFINITION 1.28. A fixed point \bar{x} of (1.17) is hyperbolic if $Df(\bar{x})$ has no eigenvalues with absolute value equal to one.

If \bar{x} is a hyperbolic fixed point of (1.17), then it is asymptotically stable if all eigenvalues of $Df(\bar{x})$ lie inside the unit disc, and unstable if some eigenvalue lies outside the unit disc.

The behavior of even one-dimensional discrete dynamical systems may be complicated and difficult to analyze.

EXAMPLE 1.29. The biologist May (1976) drew attention to the fact that the logistic map,

$$x_{n+1} = \mu x_n (1 - x_n),$$

leads to a discrete dynamical system with remarkably intricate behavior, even though the corresponding continuous logistic ODE

$$x_t = \mu x(1 - x)$$

is simple to analyze completely.

EXAMPLE 1.30. The Collatz function $C : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$C(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

An integer-valued solution of the dynamical system

$$x_{n+1} = C(x_n)$$

either grows by a factor close to 3 or collapses by a factor of 2, and always collapses at least once after it grows. For example, the orbit of the solution with initial data $x_0 = 9$ is

$$9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, \dots$$

According to the $3x + 1$, or Collatz, conjecture, for every initial data $x_0 \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $x_n = 1$, after which the iterates are periodic with orbit $(1, 4, 2)$. This conjecture has been verified numerically for all initial data x_0 less than or equal to 5×2^{60} , but its proof is an open problem.

EXAMPLE 1.31. Let $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ be the Riemann sphere and $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ a rational function. That is, $f(z) = p(z)/q(z)$ where p, q are polynomials, and we use the natural conventions for 0 and ∞ , e.g., $1/z = \infty$ if $z = 0$ and $1/z = 0$ if $z = \infty$. The dynamics of the iterated maps $z_{n+1} = f(z_n)$ is, in general, very complicated. Roughly speaking, the Fatou set $F \subset \bar{\mathbb{C}}$ of f consists of those points $z \in \bar{\mathbb{C}}$ with the property that all sufficiently nearby points behave similarly to z under iterations of f . (More precisely, the Fatou set F of f is the maximal open subset of $\bar{\mathbb{C}}$ such that the family of iterates $\{f^n : n \in \mathbb{N}\}$ is equicontinuous on F .) The Julia set $J = \bar{\mathbb{C}} \setminus F$ of f is the complement of the Fatou set, meaning that points in the Julia set divide points whose iterates behave differently. The Julia set of a rational

map is typically not a smooth curve such as a circle, but has an intricate fractal structure.

Discrete dynamical systems may arise directly as models; for example, in population ecology, $x_n \in \mathbb{R}^d$ might represent the population of the n th generation of d interacting species. But they also arise from continuous dynamical systems.

EXAMPLE 1.32. If Φ_t is the flow map of a continuous dynamical system with globally defined solutions, then the time-one map Φ_1 defines an invertible discrete dynamical system. The dimension of the discrete system is the same as the dimension of the continuous one.

EXAMPLE 1.33. The time-one map of a linear system of ODEs $x_t = Ax$ is

$$B = e^A.$$

Note that if λ is an eigenvalue of A , then e^λ is an eigenvalue of e^A , so eigenvalues of A in the left-half of the complex plane, with negative real part, map to eigenvalues of B inside the unit disc, and eigenvalues of A in the right-half-plane map to eigenvalues of B outside the unit disc. Thus, the stability properties of the fixed point $\bar{x} = 0$ in the continuous and discrete descriptions are consistent.

EXAMPLE 1.34. Consider a non-autonomous system for $x(t) \in \mathbb{R}^d$ that depends periodically on time with period 1, say,

$$x_t = f(x, t), \quad f(x, t+1) = f(x, t).$$

We define the corresponding Poincaré map $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$\Phi : x(0) \mapsto x(1).$$

Then Φ defines an autonomous discrete dynamical system of dimension d , which is one less than the dimension $d+1$ of the original system when it is written in autonomous form. This reduction in dimension makes the dynamics of the Poincaré map easier to visualize than that of the original flow, especially when $d=2$. Moreover, by continuous dependence, trajectories of the original system remain arbitrarily close over the entire time-interval $0 \leq t \leq 1$ if their initial conditions are sufficiently close, so replacing the full flow map by the Poincaré map does not lead to any essential loss of qualitative information.

Fixed points of the Poincaré map correspond to periodic solutions of the original system, although their minimal period need not be one; for example any solution of the original system with period $1/n$ where $n \in \mathbb{N}$ is a fixed point of the Poincaré map, as is any equilibrium solution with $f(\bar{x}, t) = 0$.

EXAMPLE 1.35. Consider the forced, damped pendulum with non-dimensionalized equation

$$x_{tt} + \delta x_t + \sin x = \gamma \cos \omega t$$

where γ , δ , and ω are parameters, measuring the strength of the damping, the strength of the forcing, and the (angular) frequency of the forcing, respectively. Or a parametrically forced oscillator (such as a swing)

$$x_{tt} + (1 + \gamma \cos \omega t) \sin x = 0.$$

Here, the Poincaré map $\Phi : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$ is defined by

$$\Phi : (x(0), x_t(0)) \mapsto (x(T), x_t(T)), \quad T = \frac{2\pi}{\omega}.$$

Floquet theory is concerned with such time-periodic ODEs, including the stability of their time-periodic solutions, which is equivalent to the stability of fixed points of the Poincaré map.

1.8. Rayleigh-Bénard convection

When a layer of fluid in a gravitational field is heated from below, one sees a sequence of bifurcations as the temperature difference across the layer increases. This phenomenon is named after the French physicist Henri Bénard, who carried out the first experiments around 1900, and the English applied mathematician Lord Rayleigh, who carried out the first stability analysis in 1916.

For small temperature differences, the fluid remains at rest and heat diffuses from bottom to top. When the temperature difference exceeds a critical value, one typically sees the onset of steady convection, in which warmer, lighter fluid rises, cools, and falls back down. At larger temperature differences, one may see time-periodic behavior arise, and for sufficiently large differences the flow becomes turbulent.

A relatively simple model for thermal convection is provided by the Boussinesq equations, which consist of the incompressible Navier-Stokes equations for the velocity and pressure of the fluid, with a suitable approximation for the buoyancy forces, and a heat equation for the temperature of the fluid. This problem has been studied extensively, both theoretically and experimentally, as a test case for bifurcations in fluid systems; it is also significant for engineering applications as well as in the modeling of thermal convection in the earth's atmosphere and mantle.

If $\vec{u}(\vec{x}, t)$, $p(\vec{x}, t)$, and $\theta(\vec{x}, t)$ are the fluid velocity, pressure, and temperature, respectively, as functions of space \vec{x} and time t , then the Boussinesq equations are

$$\begin{aligned}\vec{u}_t + \vec{u} \cdot \nabla \vec{u} &= -\frac{1}{\rho_0} \nabla p + \nu \Delta \vec{u} + \alpha g (\theta - \theta_1) \vec{k}, \\ \nabla \cdot \vec{u} &= 0, \\ \theta_t + \vec{u} \cdot \nabla \theta &= \kappa \Delta \theta.\end{aligned}$$

The first two equations are the incompressible Navier-Stokes equations for (\vec{u}, p) , and the third equation is an advection-diffusion equation for θ . The constant ρ_0 is a typical density of the fluid, ν is the kinematic viscosity, and κ is the thermal diffusivity.

The term $\alpha g (\theta - \theta_1) \vec{k}$ on the right-hand side of the Navier-Stokes equations describes the gravitational buoyancy force on the warmer, lighter fluid. Here, \vec{k} is the unit upward vector in the z -direction, g is the acceleration due to gravity, and α is the thermal expansion coefficient of the fluid, i.e., the increase in volume of a unit volume of fluid per unit increase in temperature. The Boussinesq approximation (which is not entirely systematic) consists of including the resulting small changes in the fluid density in the gravitational buoyancy force, where they are crucial in driving the fluid convection, and neglecting them in the inertia terms, where they typically have little effect relative to the unperturbed density.

We need to supplement the Boussinesq equations with appropriate initial and boundary conditions. The initial conditions are

$$\vec{u}(\vec{x}, 0) = \vec{u}_0(\vec{x}), \quad \theta(\vec{x}, 0) = \phi_0(\vec{x}),$$

where \vec{u}_0 and ϕ_0 are given initial velocity and temperature fields. It is not necessary to impose initial conditions for the pressure p since the Navier-Stokes equations do not prescribe the time-derivative p_t .

One can impose different kinds of boundary conditions. For simplicity, we consider convection in a two-dimensional box of length ℓ and height h , and write the spatial variables as $\vec{x} = (x, z)$ and the fluid velocity as $\vec{u} = (u, w)$. We suppose that the top of the box ($z = h$) is maintained at temperature θ_1 and the bottom of the box ($z = 0$) is maintained at temperature $\theta_0 > \theta_1$. One possible set of boundary conditions, corresponding to a rigid box with insulated sides, is

$$\begin{aligned} \vec{u} &= 0 && \text{on } x = 0, \ell \text{ and } z = 0, h, \\ \theta &= \theta_0 && \text{on } z = 0, \quad \theta = \theta_1 && \text{on } z = h, \\ \theta_x &= 0 && \text{on } x = 0, \ell. \end{aligned}$$

An alternative boundary condition for the velocity is $u_z = w = 0$ on $z = 0, h$. This boundary condition corresponds to a boundary that is allowed to slip freely in the x -direction under tangential viscous stresses in the fluid. It is less physically applicable than the ‘no-slip’ condition $\vec{u} = 0$, but it is often used because it simplifies the analysis of the resulting initial-boundary value problem.

This problem depends on several different dimensional parameters. The only parameter involving the dimension of mass is the density ρ_0 , which can be removed by defining a rescaled pressure $\tilde{p} = p/\rho_0$, and only temperature differences are significant. This leaves us with 6 parameters from the equations, boundary conditions, and the geometry of domain:

$$\nu, \quad \kappa, \quad \alpha g, \quad \delta\theta = \theta_1 - \theta_0, \quad h, \quad \ell.$$

These parameters depend on 3 dimensions: length (L), time (T), and temperature (Θ). Denoting the dimensions of a parameter c by $[c]$, we have

$$[\nu] = [\kappa] = \frac{L^2}{T}, \quad [\alpha g] = \frac{L}{\Theta T^2}, \quad [\delta\theta] = \Theta, \quad [h] = [\ell] = L.$$

The dimensions of the diffusivities ν, κ can be read off from the equations, since every term appearing in the same equation must have the same dimension, e.g.,

$$\frac{[\nu]}{L^2} = [\nu\Delta] = [\partial_t] = \frac{1}{T}.$$

We can construct $3 = 6 - 3$ independent dimensionless parameters from 6 dimensional parameters depending on 3 dimensions. One possible choice is

$$Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{\alpha g \cdot \delta\theta \cdot h^3}{\nu\kappa}, \quad \beta = \frac{h}{\ell}.$$

The Prandtl number Pr measures the relative diffusivity of momentum and heat in the fluid, the Rayleigh number Ra is a dimensionless temperature difference, and β is the aspect ratio of the fluid layer. The behavior of the system is independent of the choice of units and is determined by the values of these dimensionless parameters. The choice of dimensionless parameters is not unique; for example, instead of the Rayleigh number, one could use the Grashof number

$$Gr = \frac{Ra}{Pr} = \frac{\alpha g \cdot \delta\theta \cdot h^3}{\nu^2}.$$

The values of these dimensionless parameters may vary over a wide range in different systems. For example, most early experiments (such as the original ones

of Bénard) were done in thin fluid layers with aspect ratios close to 0, but Ahlers and Beringer (1978) were able to carry out convection experiments in liquid helium with aspect ratios up to 57. The Prandtl number of water at standard conditions is approximately 7, while the Prandtl number of the highly viscous rocks in the earth's mantle is of the order 10^{25} .

Lorenz derived his system of ODEs (1.2) by truncating the Rayleigh-Bénard equations to equations for three Fourier modes. He looked for solutions of the form

$$\begin{aligned} u(x, z, t) &= X(t) \sin\left(\frac{\pi x}{\ell}\right) \cos\left(\frac{\pi z}{h}\right), \\ w(x, z, t) &= X(t) \cos\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{\pi z}{h}\right), \\ \theta(x, z, t) &= \theta_e(z) + Y(t) \cos\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{\pi z}{h}\right) + Z(t) \sin\left(\frac{2\pi z}{h}\right), \end{aligned}$$

where

$$\theta_e(z) = \theta_0 + (\theta_1 - \theta_0) \frac{z}{h}$$

is the equilibrium temperature profile in a fluid at rest. These functions satisfy the incompressibility condition $u_x + w_z = 0$ and 'free' boundary conditions. Lorenz then required that appropriate lowest-order Fourier components of the Boussinesq equations are satisfied, and, after rescaling (X, Y, Z) to (x, y, z) , he obtained (1.2). The parameters σ , r , and b in the Lorenz equation correspond to Pr , Ra , and β , respectively, and the equilibrium $(x, y, z) = (0, 0, 0)$ corresponds to the state in which the fluid is at rest.

1.9. References

For introductory discussions of the theory of ODEs see [1, 11]. Detailed accounts of the mathematical theory are given in [3, 9, 13]. A general introduction to nonlinear dynamical systems, with an emphasis on applications, is in [12]. An introduction to bifurcation theory in continuous and discrete dynamical systems is [7]. For a rigorous but accessible introduction to chaos in discrete dynamical systems, see [4]. A classic book on nonlinear dynamical systems is [8].