Methods of Applied Mathematics
Math 207A, Fall 2014
Final Solutions

1. [25%] Consider the following scalar ODE for $x(t)$, depending on a parameter $-\infty < \mu < \infty$,

$$x_t = x^2 - \mu^2 x.$$

Find the equilibria and the bifurcation value. Draw phase lines for various values of μ and draw the bifurcation diagram.

Solution

- The equilibria satisfy $x^2 - \mu^2 x = 0$, so $x = 0$ or $x = \mu^2$. Writing $f(x;\mu) = x^2 - \mu^2 x$, we have $f_x(x;\mu) = 2x - \mu^2$, so $f_x(0;\mu) < 0$ and $f_x(\mu^2, \mu) = \mu^2 > 0$ for $\mu \neq 0$. It follows that $x = 0$ is asymptotically stable and $x = \mu^2$ is unstable. For $\mu = 0$, there is one semi-stable, nonyperbolic equilibrium at $x = 0$.

- Phase lines and the bifurcation diagram are shown in Figure 1.

2. [25%] (a) Explain why the following Lotka-Volterra equations provide a reasonable model of the interaction between a prey (e.g., rabbits) with population $x(t) \geq 0$ and a predator (e.g., foxes) with population $y(t) \geq 0$:

$$x_t = x(1-y), \qquad y_t = (x-1)y.$$

(b) Find the equilibria, linearize around each one, and classify them.

(c) Integrate the first-order ODE for dy/dx and show that the orbits are given by the implicit equation $f(x) + f(y) = \text{constant}$ for a suitable function $f(x)$. Sketch the graph of f, and use this result to sketch the phase plane of the Lotka-Volterra equations. How do solutions of this system behave in time?

Solution

- (a) In the absence of predators ($y = 0$), the prey population grows exponentially in time ($x_t = x$), and in the absence of prey ($x = 0$), the predators have no food and the predator population decays exponentially in time ($y_t = -y$). The growth rate of the prey switches from positive to negative when the predator population becomes too large ($y > 1$), and the growth rate of the predators switches from negative to positive when there is enough prey ($x > 1$).

- Alternatively, one can interpret the ODE $x_t = x - xy$ as a linear growth rate of the prey minus the rate at which predators catch the prey, with the probability of a predator prey encounter being proportional to xy, and analogously for $y_t = -y + xy$.

- (b) The equilibria are $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.

- The linearization about $(0, 0)$ is

  $$x_t = x, \qquad y_t = -y,$$

  with eigenvalues $\lambda = \pm 1$, so $(0, 0)$ is a saddle point. The unstable direction is $(1, 0)$ and the stable direction is $(0, 1)$.

- Writing $x = 1 + x'$, $y = 1 + y'$, we find that The linearization about $(1, 1)$ is

  $$x'_t = -y', \qquad y'_t = x',$$

  with eigenvalues $\lambda = \pm i$, so $(0, 0)$ is a center.

- We have $dy/dx = y_t/x_t$ or

$$\frac{dy}{dx} = \frac{(x-1)y}{x(1-y)}.$$

Separating variables in this ODE and integrating the result, we get

$$\int \frac{1-y}{y}\, dy = \int \frac{x-1}{x}\, dx,$$

or (assuming that $x, y > 0$)

$$f(x) + f(y) = C, \qquad f(x) = x - \log x.$$

- A plot of $f(x)$ is shown in Figure 2. It is a convex function with a minimum at $x = 1$, and $f(x) \to +\infty$ as $x \to 0^+$ and $x \to +\infty$. It follows that the level sets of $f(x) + f(y)$ are closed curves surrounding the equilibrium $(1, 1)$. The x and y axes are invariant manifolds for the flow, so we get the phase plane shown in Figure 2.

- The solutions are time-periodic and oscillate around the stable, but not asymptotically stable, equilibrium $(1, 1)$. For example, if there are initially a lot of rabbits and only a few foxes, then the foxes have lots of food, so the fox population grows. When there are enough foxes, the rabbit population declines. After a while, there aren't enough rabbits to support the foxes, so the fox population declines, and this allows the rabbit population to recover. The system is then back in a state with lots of rabbits and a few foxes.

3 [25%] (a) Define what it means for an equilibrium $\bar{x}$ of a dynamical systems $x_t = f(x)$ to be: (i) (Liapounov) stable; (ii) asymptotically stable.

(b) Write the 2×2 system

$$x_t = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}},$$
$$y_t = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}},$$

in polar coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$, show that the equilibria are $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$, and use the result to sketch the (x, y)-phase plane of the system.

(c) Show that there is a neighborhood U of the equilibrium $(x, y) = (1, 0)$ such that $\Phi(t)(x_0, y_0) \to (1, 0)$ as $t \to +\infty$, where $\Phi(t) : \mathbb{R}^2 \to \mathbb{R}^2$ is the flow of the system. Is $(1, 0)$ a stable equilibrium?

Solution

- (a) (i) The equilibrium $\bar{x}$ is stable if for every neighborhood U of $\bar{x}$, there exists a neighborhood $V \subset U$ of $\bar{x}$ such that $\Phi(t)(x_0) \in U$ for all $t > 0$ whenever $x_0 \in V$, where $\Phi(t)$ is the flow map of the dynamical system. More informally, solutions that are sufficiently close to $\bar{x}$ remain close to $\bar{x}$.

- (a) (ii) The equilibrium is asymptotically stable if it is stable and there exists a neighborhood U of $\bar{x}$ such that $\Phi(t)(x_0) \to \bar{x}$ as $t \to +\infty$ whenever $x_0 \in U$.

- (b) Using the ODEs and simplifying the result, we compute that

$$r_t = \frac{x x_t + y y_t}{r} = r - r^3,$$
$$\theta_t = \frac{x y_t - y x_t}{r^2} = 1 - \cos\theta.$$

  The equilibria are $r = 0$ and $(r, \theta) = (1, 0)$, corresponding to $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$.

- The phase lines for the scalar r and θ equations are shown in Figure 3. For any $r(0) \neq 0$, we have $r(t) \to 0$ as $t \to +\infty$, while $\theta(t)$ moves counterclockwise round the circle and $\theta(t) \to 0$ as $t \to +\infty$. This gives a phase plane like the one sketched in Figure 3. The stable manifold of $(1, 0)$ is the positive x-axis, while the center manifold is the circle $r = 1$.

- Every solution with initial value in the neighborhood U shown in the figure approaches $(1, 0)$ as $t \to +\infty$, but there is no neighborhood V of $(1, 0)$ such that solutions remain in U for all $t \geq 0$ whenever the initial values are in V, so the equilibrium is not stable. This is because the orbits make a large excursion into the left-half plane before re-entering the neighborhood U.

4. [25%] The KPP equation

$$u_t = u_{xx} + u\,(1-u)$$

describes the diffusion of a spatially distributed species with logistic growth, where $u(x,t)$ is the (nondimensionalized) population of the species at spatial location x and time t. Traveling wave solutions of the KPP equation, given by $u(x,t) = f(x-ct)$, satisfy the ODE

$$f'' + cf' + f(1-f) = 0. \tag{1}$$

(a) Assume that the wave speed $c > 0$ is positive. Find the equilibria of this ODE. Linearize (1) about the equilibra and classify them, depending on c.

(b) Give a physical interpretation of (1) as an ODE for a damped, conservative system. What is the corresponding potential $V(f)$?

(c) Sketch the phase plane of (1) in appropriate ranges of the wave speed parameter $c > 0$. For what values of c are there nonnegative, bounded traveling wave solutions? Give a qualitative sketch of the graph of $f(z)$ versus z for one of these waves. What is the biological interpretation of these solutions?

Solution

- (a) The equilibria are $(f, f') = (0,0)$ and $(f, f') = (1,0)$.
- The linearization of (1) about $(0,0)$ is

  $$f'' + cf' + f = 0,$$

  with characteristic equation $\lambda^2 + c\lambda + 1 = 0$, and roots

  $$\lambda = \frac{-c \pm \sqrt{c^2-4}}{2}.$$

  If $0 < c < 2$, then we have a complex conjugate pair of eigenvalues, with negative real part, and $(0,0)$ is a stable spiral point. If $c \geq 2$, then we have two real and negative eigenvalues (repeated if $c = 2$) and $(0,0)$ is a stable node.

- The linearization of (1) about $(1,0)$ is
$$f'' + cf' - f = 0,$$
with characteristic equation $\lambda^2 + c\lambda - 1 = 0$, and roots
$$\lambda = \frac{-c \pm \sqrt{c^2+4}}{2}.$$
There are two real eigenvalues of opposite signs, so $(1,0)$ is a saddle point.

- (b) Formally, the ODE is the same as the ODE for a linearly damped conservative system, with damping constant c,
$$f'' + cf' + \frac{dV(f)}{df} = 0, \qquad V(f) = \frac{1}{2}f^2 - \frac{1}{3}f^3,$$
where we interpret the independent variable as time. This mechanical analogy is useful in visualizing the phase plane (see Figure 4).

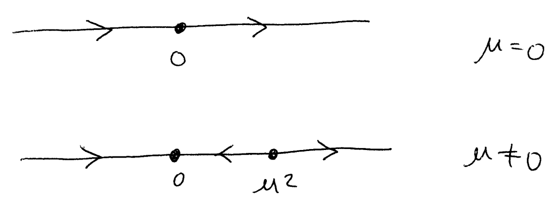
- (c) There is a heteroclinic orbit $f(z)$ consisting of the unstable manifold of the saddle point $(1,0)$ that is attracted to the stable spiral/node $(0,0)$, so $f(z) \to 1$ as $z \to -\infty$ and $f(z) \to 0$ as $z \to +\infty$. In the interpretation of (b), this orbits falls off the unstable maximum of $V(f)$ at $f = 1$ and into the potential well around the stable minimum of $V(f)$ at $f = 0$. The solution is underdamped if $c < 2$, when f is oscillates around the equilibrium $f = 0$, and then f is negative in the neighborhood of a spiral point. We only get a nonnegative, bounded traveling wave when $c \geq 2$, corresponding to an overdamped oscillator.

- This travelling wave corresponds to the invasion of an unpopulated region $u = 0$ for large positive x by a population $u = 1$ for large negative x.

- For a more detailed analysis of this traveling wave solution, see e.g., Lecture 1, Section 3 of these notes:

  `https://www.math.ucdavis.edu/~hunter/m280_09/applied_math.html`

1. Phase lines



Bifurcation diagram

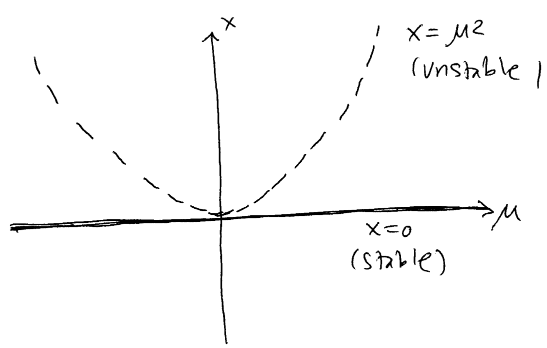


Figure 1: Phase lines and bifurcation diagram for Problem 1.

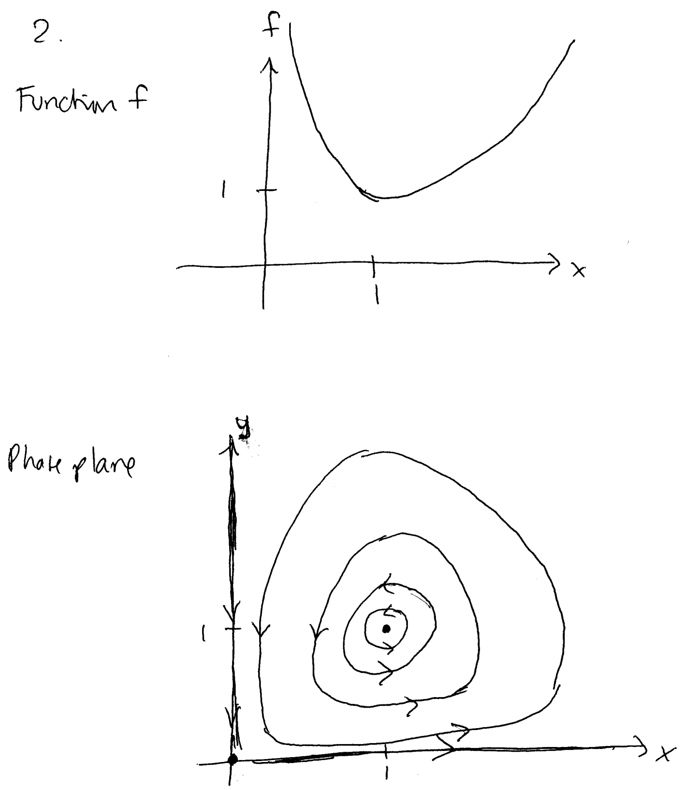
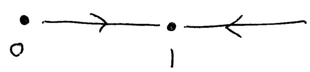
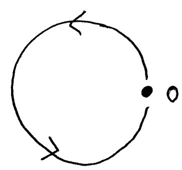


Figure 2: Plot of f and phase plane for Problem 2.

3. $r_t = r - r^3$



$\theta_t = 1 - \cos\theta$



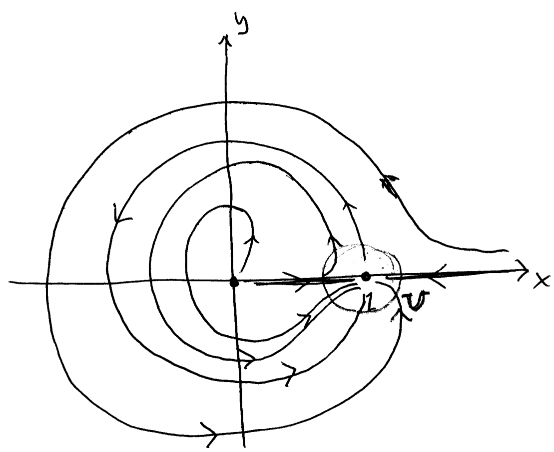


Figure 3: Phase plane for Problem 3.

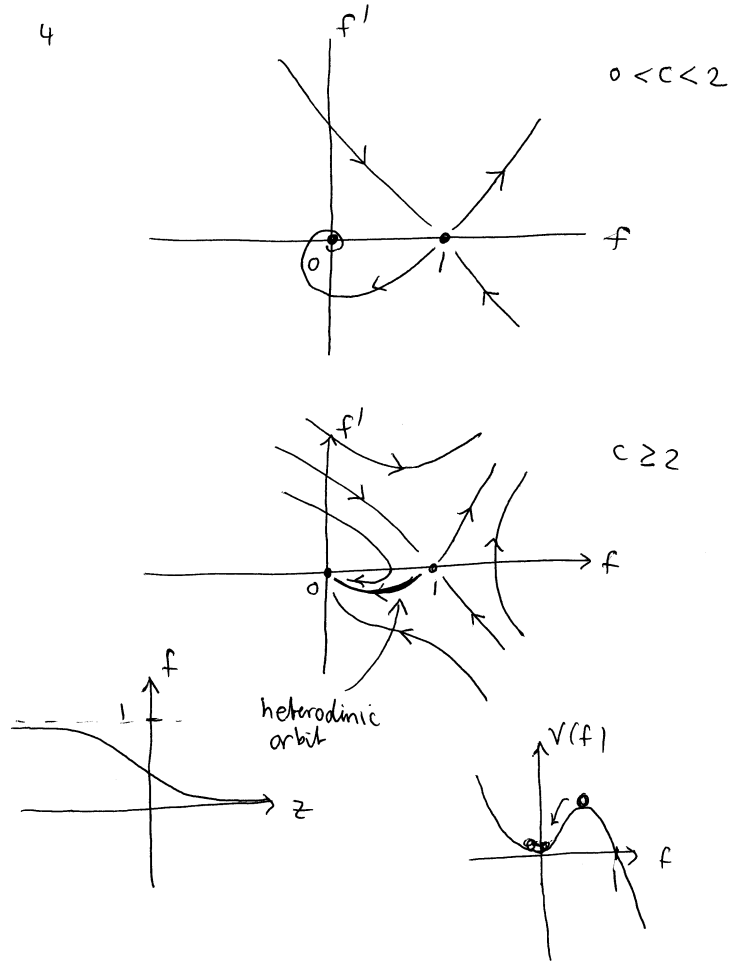


Figure 4: Phase planes for Problem 4.