Methods of Applied Mathematics
Math 207A, Fall 2014
Midterm: Solutions

1 [25%] (a) Solve the scalar initial value problem

$$\dot{x} = e^x,$$
$$x(0) = x_0,$$

for $x(t)$, determine the maximal t-interval of existence of the solution, and write an expression for the flow map $\Phi(t) : x_0 \mapsto x(t)$.
(b) Verify explicitly that, where defined, the flow map you found in (a) has the group property $\Phi(s) \circ \Phi(t) = \Phi(s+t)$.

Solution

- (a) Separating variables and integrating, we get

$$\int e^{-x}\, dx = \int dt + C$$
$$-e^{-x} = t + C.$$

The initial condition gives $C = -e^{-x_0}$, and solving for x we get that

$$x(t) = -\log\left(e^{-x_0} - t\right).$$

- The maximal interval of existence is $-\infty < t < T(x_0)$ where

$$T(x_0) = e^{-x_0} > 0.$$

The time-t flow map is given by

$$\Phi(t)(x_0) = -\log\left(e^{-x_0} - t\right).$$

(b) We have

$$\begin{aligned}\Phi(s) \circ \Phi(t)(x_0) &= \Phi(s)\left(-\log\left(e^{-x_0} - t\right)\right)\\ &= -\log\left(e^{\log\left(e^{-x_0} - t\right)} - s\right)\\ &= -\log\left(e^{-x_0} - t - s\right)\\ &= \Phi(s+t)(x_0).\end{aligned}$$

2 [25%] Suppose that $x(t)$ satisfies the second-order, scalar ODE

$$\ddot{x} + \beta\dot{x} + xe^{x^2} = 0 \tag{1}$$

where $\beta \geq 0$ is a constant, and define $E(t)$ by

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}e^{x^2}.$$

(a) Show that $\dot{E} \leq 0$.

(b) What can you say about the global existence of solutions of the initial value problem for (1): (i) forward in time; (ii) backward in time?

(c) Give a physical interpretation of (1) and of E.

Solution

- (a) Using the chain rule and the ODE, we get

$$\dot{E} = \dot{x}\ddot{x} + xe^{x^2}\dot{x} = \dot{x}\left(\ddot{x} + xe^{x^2}\right) = -\beta\dot{x}^2 \leq 0.$$

  (b) (i) From (a), $E(t)$ is a decreasing function of t, so $E(t) \leq E(0)$ for $t \geq 0$. Since $E \to \infty$ as $|(x, \dot{x})| \to \infty$, it follows that $(x, \dot{x})$ remains bounded, and the extension theorem implies that the solution exists for all $t \geq 0$.

- (b) (ii) We can't immediately conclude that the solution exists for all $t < 0$, since $E(t) \geq E(0)$ for $t < 0$, and this inequality doesn't rule out the possibility that $|(x, \dot{x})| \to \infty$ in finite negative time.

- (c) This ODE describes the one-dimensional motion of a particle with unit mass and position $x(t)$ at time t in a conservative force field $F = -V_x$, with potential $V(x) = e^{x^2}/2$. The particle is subject to linear damping proportional to its velocity. The function E is the energy of the particle (kinetic + potential).

3 [25%] (a) Draw the phase line for the ODE $x_t = \sin x$.
(b) Describe qualitatively what bifurcations occur for the scalar ODE

$$x_t = \mu x + \sin x$$

as the parameter $\mu \geq 0$ is increased from 0. Sketch the bifurcation diagram.

Solution

- (a) There are infinitely many hyperbolic equilibria at $x = n\pi$ for $n \in \mathbb{Z}$. Writing $f(x) = \sin x$, we have $f'(n\pi) = \cos n\pi = (-1)^n$, so the equilibria are stable if n is odd ($f' < 0$) and unstable if n is even ($f' > 0$). See Figure 2 for the phase line.

- (b) For $\mu > 0$, there are only finitely many equilibria, given by the intersections of the line $y = \mu x$ with $y = -\sin x$ (see Figure 1). Pairs of stable and unstable equilibria are annihilated in successive saddle-node bifurcations as μ increases from 0.

- The saddle-node bifurcations occur at $(x, \mu) = (\pm x_n, \mu_n)$ where

  $$\{(x_n, \mu_n) : n = 0, 1, 2, \dots\}$$

  are solutions of the transcendental (and not explicitly solvable) equations $f(x, \mu) = 0$, $f_x(x, \mu) = 0$, or

  $$\mu x + \sin x = 0, \qquad \mu + \cos x = 0,$$

  with $\mu_0 > \mu_1 > \mu_2 > \cdots > 0$ and $\mu_n \downarrow 0$ as $n \to \infty$. See Figure 2 for the bifurcation diagram.

- For $\mu > \mu_0$, where $0 < \mu_0 < 1$ is the parameter value at the last saddle-node bifurcation (numerically, $\mu_0 \approx 0.2172$), there is only one unstable equilibrium at $x = 0$.

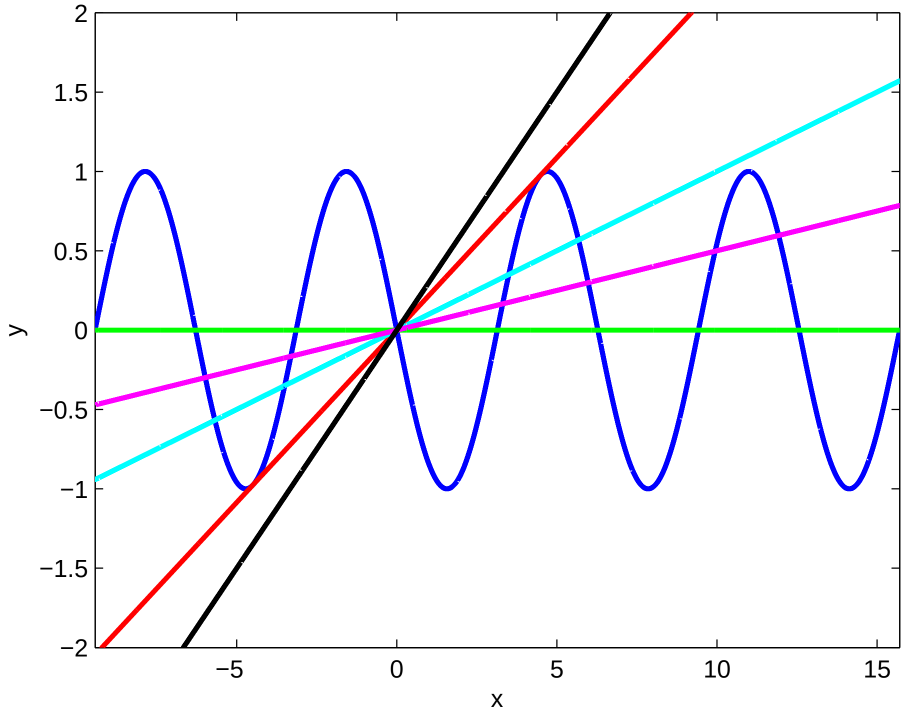

Figure 1: Plot of the graphs $y = \sin x$ (in blue) and $y = \mu x$ (in colors) for $\mu = 0, 0.05, 0.1, 0.2172, 0.3$.

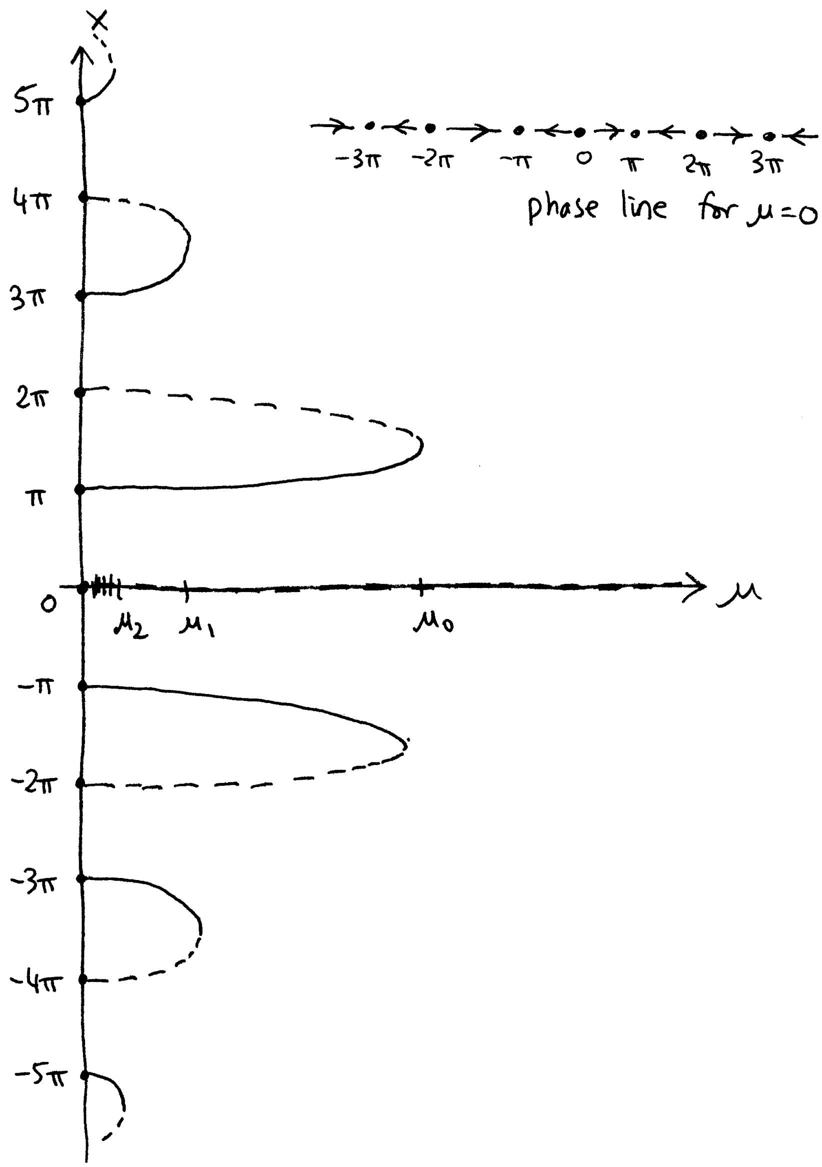

Figure 2: Phase line of $\dot{x} = \mu x + \sin x$ for $\mu = 0$ and its bifurcation diagram for $\mu \geq 0$.

4 [25%] Suppose that x, y satisfy the following 2×2 system of ODEs

$$\begin{aligned} \dot{x} &= ax + y + \mu x(x^2 + y^2), \\ \dot{y} &= ay - x + \mu y(x^2 + y^2), \end{aligned} \tag{2}$$

where $-\infty < a, \mu < \infty$ are constant parameters.
(a) Show that $r = \sqrt{x^2 + y^2}$ satisfies the scalar ODE

$$\dot{r} = ar + \mu r^3. \tag{3}$$

(b) Draw the phase lines of (3) in $r \geq 0$ for: (i) $a, \mu > 0$; (ii) $a < 0$, $\mu > 0$; (iii) $a > 0$, $\mu < 0$; (iv) $a, \mu < 0$.
(c) State how the stability of the equilibrium $(x, y) = (0, 0)$ of (2) depends on (a, μ).

Solution

- (a) Using the chain rule and the ODEs, we compute that (for $r > 0$)

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ &= \frac{ax^2 + \mu x^2(x^2 + y^2) + ay^2 + \mu y^2(x^2 + y^2)}{r} \\ &= \frac{ar^2 + \mu r^4}{r} \\ &= ar + \mu r^3. \end{aligned}$$

- (b) If a, μ have the same sign, then the only equilibrium of the ODE in $r \geq 0$ is $r = 0$. If a, μ have opposite signs, then there is an additional equilibrium at

$$r = r_0(a, \mu), \qquad r_0(\mu, a) = \sqrt{-\frac{a}{\mu}}.$$

- Writing

$$f(r, \mu, a) = ar + \mu r^3, \qquad f_r(r, \mu, a) = a + 3\mu r^2,$$

  we have $f_r(0, \mu, a) = a$, so the equilibrium $r = 0$ is asymptotically stable if $a < 0$ and unstable if $a > 0$. Also, we have $f_r(r_0, \mu, a) = -2a$, so the equilibrium $r = r_0$ is asymptotically stable if $a > 0$ ($f_r < 0$) and unstable if $a < 0$ ($f_r > 0$). These considerations give the phases lines shown in Figure 3. Alternatively, you can look at the sign of $f(r, \mu, a)$ as a function of r to determine where $r(t)$ is increasing or decreasing.

- (c) Since $r = |(x, y)|$, the stability of the equilibrium $(x, y) = (0, 0)$ for (2) is the same as the stability of the equilibrium $r = 0$ for (3).
- If $a \neq 0$, then the equilibrium $r = 0$ is hyperbolic, so it is asymptotically stable for $a < 0$ and unstable for $a > 0$, whatever the value of μ.
- If $a = 0$, then the non-hyperbolic equilibrium $r = 0$ of $\dot{r} = \mu r^3$ is asymptotically stable if $\mu < 0$, stable but not asymptotically stable if $\mu = 0$, and unstable if $\mu > 0$.

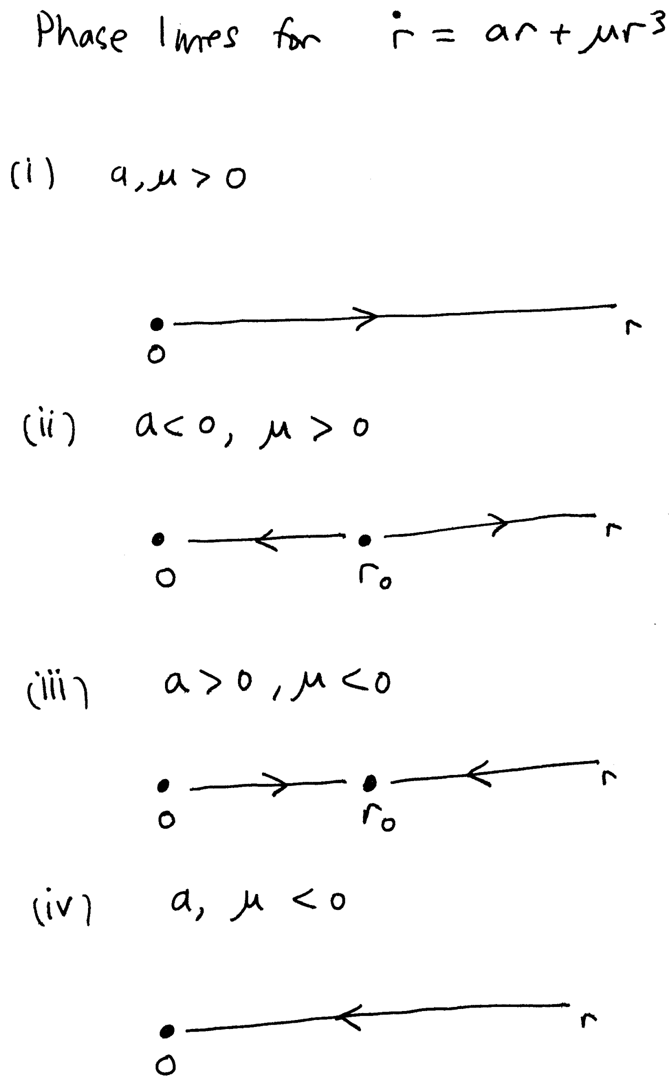

Figure 3: Phase lines of $\dot{r} = ar + \mu r^3$.