Problem set 1

Math 207A, Fall 2014

1. Mathieu's equation for $x(t) \in \mathbb{R}$ is

$$x_{tt} + (a - 2q\cos 2t)\,x = 0,$$

where a, q are constant parameters. This ODE describes a simple-harmonic oscillator whose frequency varies periodically in time. Write Mathieu's equation as a first-order autonomous system. Is Mathieu's equation linear? Is the corresponding autonomous first-order system linear?

2. Solve the IVP with $x(0) = x_0$ for the following scalar ODEs:

$$\text{(a) } x_t = x^{1/3}; \qquad \text{(b) } x_t = x^3; \qquad \text{(c) } x_t = \frac{x^3}{1+x^2}.$$

Discuss the existence (local/global) and uniqueness of solutions in each case.

3. A gradient system for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is a system of the form

$$\dot{x} = -\nabla V(x), \qquad \text{or} \quad \dot{x}_i = -\frac{\partial V}{\partial x_i} \quad (1 \le i \le n),$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and ∇ is the gradient with respect to x.

(a) If $x(t)$ is a solution of this gradient system with $x(0) = x_0$, show that $V(x(t)) \le V(x_0)$ for all $t \ge 0$.

(b) Show that the following system for $(x, y) \in \mathbb{R}^2$ is a gradient system

$$\dot{x} = -x + 2y - x^3, \qquad \dot{y} = 2x - y - y^3,$$

and deduce that solutions of the initial value problem exist for all $t \ge 0$. Do solutions necessarily exist for all $t < 0$?

4. Write a MATLAB script to solve the Lorentz equations

$$\dot{x} = s(-x + y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. Use Lorenz's parameter values $s = 10$, $r = 28$, $b = 8/3$ to compute the following solutions. Submit a copy of your script and the two plots.

(a) Plot the trajectory for initial data $(x_0, y_0, z_0) = (0, 1, 0)$ as a parametric curve in (x, y, z)-phase space for $0 \le t \le 30$.

(b) Plot, on the same graph, the solutions for $x(t)$ with $0 \le t \le 30$ and the two sets of initial data: (i) $(x_0, y_0, z_0) = (0, 1, 0)$; (ii) $(x_0, y_0, z_0) = (0, 1.01, 0)$.