Problem set 1: Solutions
Math 207A, Fall 2014

1. Mathieu's equation for $x(t) \in \mathbb{R}$ is

$$x_{tt} + (a - 2q\cos 2t)\, x = 0,$$

where a, q are constant parameters. This ODE describes a simple-harmonic oscillator whose frequency varies periodically in time. Write Mathieu's equation as a first-order autonomous system. Is Mathieu's equation linear? Is the corresponding autonomous first-order system linear?

Solution

- To write the ODE as an antonomous first order system, we introduce new variables $y = x_t$, $s = t$. Then (x, y, s) satisfies the 3×3 autonomous system

$$\begin{aligned} x_t &= y, \\ y_t &= -\,(a - 2q\cos 2s)\, x, \\ s_t &= 1. \end{aligned}$$

- When regarded as a first-order autonomous system, this system is nonlinear (because of the term $\cos 2s \cdot x$), even though Mathieu's equation is linear.

2. Solve the IVP with $x(0) = x_0$ for the following scalar ODEs:

$$\text{(a) } x_t = x^{1/3}; \qquad \text{(b) } x_t = x^3; \qquad \text{(c) } x_t = \frac{x^3}{1+x^2}.$$

Discuss the existence (local/global) and uniqueness of solutions in each case.

Solution

- (a) The existence-uniqueness theorem implies that a solution has a unique local continuation so long as $x(t) \neq 0$, since $x^{1/3}$ is continuously differentiable for $x \neq 0$. Moreover, if $x(t_0) = 0$, then the only way to continue the solution backward in time for $t < t_0$ is by $x(t) = 0$, since

$$\frac{d}{dt}(x^2) = 2x^{4/3} \geq 0,$$

so $x^2(t)$ decreases as t decreases and $x(t)$ cannot be non-zero for $t < t_0$ if $x(t_0) = 0$.

- Solving the ODE by separating variables, we get

$$\begin{aligned} \int \frac{dx}{x^{1/3}} &= \int dt + C, \\ \frac{3}{2}x^{2/3} &= t + C, \\ x(t) &= \left[\frac{2}{3}(t+C)\right]^{3/2}. \end{aligned}$$

- It follows that if $x_0 \neq 0$, then there is a unique global solution

$$x(t) = \begin{cases} (\operatorname{sgn} x_0)[2(t-t_0)/3]^{3/2} & \text{if } t > t_0, \\ 0 & \text{if } t \leq t_0, \end{cases}$$

where $t_0 = -3x_0^{2/3}/2 < 0$, and

$$\operatorname{sgn} x_0 = \begin{cases} 1 & \text{if } x_0 > 0, \\ -1 & \text{if } x_0 < 0, \end{cases}$$

is the sign of x_0. Note that this function is continuously differentiable and satisfies the ODE for all $t \in \mathbb{R}$; in particular, $\dot{x}(t_0) = 0$.

- If $x_0 = 0$, then there are non-unique, global solutions. Either $x(t) = 0$ for all $t \in \mathbb{R}$, or

$$x(t) = \begin{cases} \pm[2(t - t_0)/3]^{3/2} & \text{if } t > t_0, \\ 0 & \text{if } t \le t_0, \end{cases}$$

where $t_0 \ge 0$ is an arbitrary constant. This non-uniqueness is possible because $f(x) = x^{1/3}$ is not a Lipschitz continuous function of x at $x = 0$.

- (b) The function $f(x) = x^3$ is (locally) Lipschitz continuous on $\mathbb{R}$, so a unique local solution exists for all $x_0 \in \mathbb{R}$.

- If $x_0 \ne 0$, then separation and integration of the ODE shows that there is a unique local solution

$$x(t) = \frac{x_0}{\sqrt{1 - 2x_0^2 t}}, \qquad -\infty < t < \frac{1}{2x_0^2}.$$

This solution exists globally backward in time, but only locally forward time. If $x_0 = 0$, then there is a unique global solution $x(t) = 0$.

- (c) The function $f(x) = x^3/(1 + x^2)$ is (locally) Lipschitz continuous on $\mathbb{R}$, so a unique local solution exists for all $x_0 \in \mathbb{R}$.

- If $x_0 = 0$, then the unique solution is $x(t) = 0$. If $x_0 \ne 0$, then separation and integration of the ODE gives

$$\log|x| - \frac{1}{2x^2} = t + \log|x_0| - \frac{1}{2x_0^2}.$$

We can't solve this transcendental equation for $x(t)$ explicitly, but it defines a global solution of the ODE.

- The existence of a global solution is seen most easily from Gronwall's inequality:

$$\frac{d}{dt}(x^2) = \frac{2x^4}{1 + x^2} \le 2x^2,$$

which implies that $|x(t)| \le |x_0|e^t$, so solutions remain bounded and exist globally in time by the extension theorem.

3. A gradient system for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is a system of the form

$$\dot{x} = -\nabla V(x), \qquad \text{or} \quad \dot{x}_i = -\frac{\partial V}{\partial x_i} \quad (1 \le i \le n),$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and ∇ is the gradient with respect to x.

(a) If $x(t)$ is a solution of this gradient system with $x(0) = x_0$, show that $V(x(t)) \le V(x_0)$ for all $t \ge 0$.

(b) Show that the following system for $(x, y) \in \mathbb{R}^2$ is a gradient system

$$\dot{x} = -x + 2y - x^3, \qquad \dot{y} = 2x - y - y^3,$$

and deduce that solutions of the initial value problem exist for all $t \ge 0$. Do solutions necessarily exist for all $t < 0$?

Solution

- (a) Using the chain rule and the ODE, we get

  $$\frac{d}{dt} V(x(t)) = \nabla V \cdot \frac{dx}{dt} = -|\nabla V|^2 \le 0,$$

  so $V(x(t))$ is a decreasing function of t, and $V(x(t)) \le V(x_0)$ for all $t \ge 0$.

- (b) The system can be written as

  $$\dot{x} = -\frac{\partial V(x, y)}{\partial x}, \qquad \dot{y} = -\frac{\partial V(x, y)}{\partial y}$$

  where

  $$V(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 2xy.$$

- Suppose that $x(0) = x_0$, $y(0) = y_0$. Since $V(x, y) \to \infty$ as $|(x, y)| \to \infty$ and $V(x(t), y(t)) \le V(x_0, y_0)$, solutions remain bounded for $t \ge 0$, so they exist for all $t \ge 0$ by the extension theorem. Explicitly, we have

  $$V(x, y) = \frac{1}{4}(x^2 - 1)^2 + \frac{1}{4}(y^2 - 1)^2 + (x - y)^2 - \frac{1}{2},$$

  and therefore, for $t \ge 0$,

  $$x^2(t), y^2(t) \le 1 + \sqrt{2 + 4V(x(t), y(t))} \le 1 + \sqrt{2 + 4V(x_0, y_0)}.$$

- For $t < 0$, we have $V(x(t), y(t)) \ge V(x_0, y_0)$, so we can't conclude that solutions remain bounded and exist for all $t < 0$.

4. Write a MATLAB script to solve the Lorentz equations

$$\dot{x} = s(-x+y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. Use Lorenz's parameter values $s = 10$, $r = 28$, $b = 8/3$ to compute the following solutions. Submit a copy of your script and the two plots.

(a) Plot the trajectory for initial data $(x_0, y_0, z_0) = (0, 1, 0)$ as a parametric curve in (x, y, z)-phase space for $0 \leq t \leq 30$.

(b) Plot, on the same graph, the solutions for $x(t)$ with $0 \leq t \leq 30$ and the two sets of initial data: (i) $(x_0, y_0, z_0) = (0, 1, 0)$; (ii) $(x_0, y_0, z_0) = (0, 1.01, 0)$.

Solution

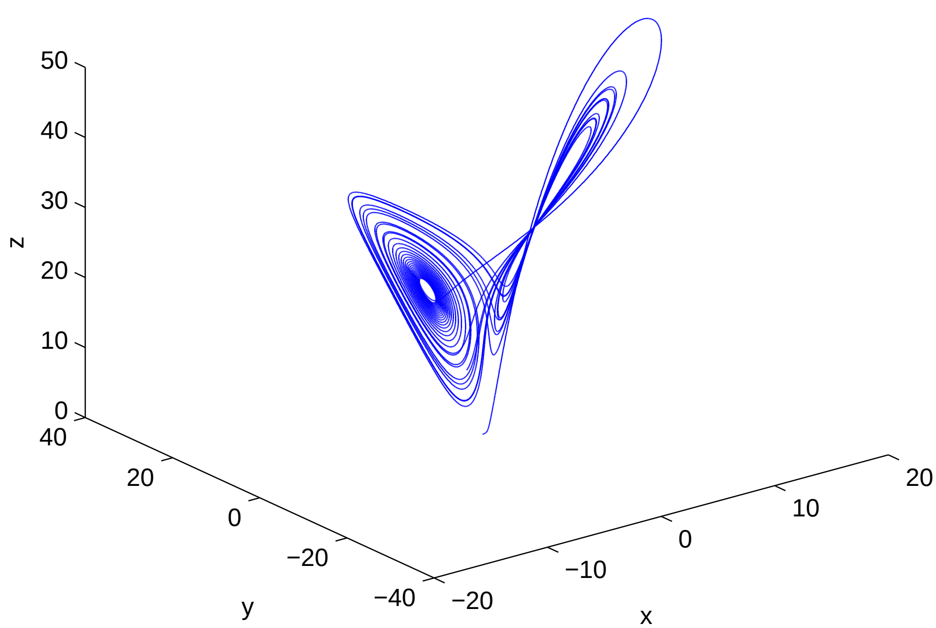

Figure 1: (a) Plot of the trajectory in phase space.

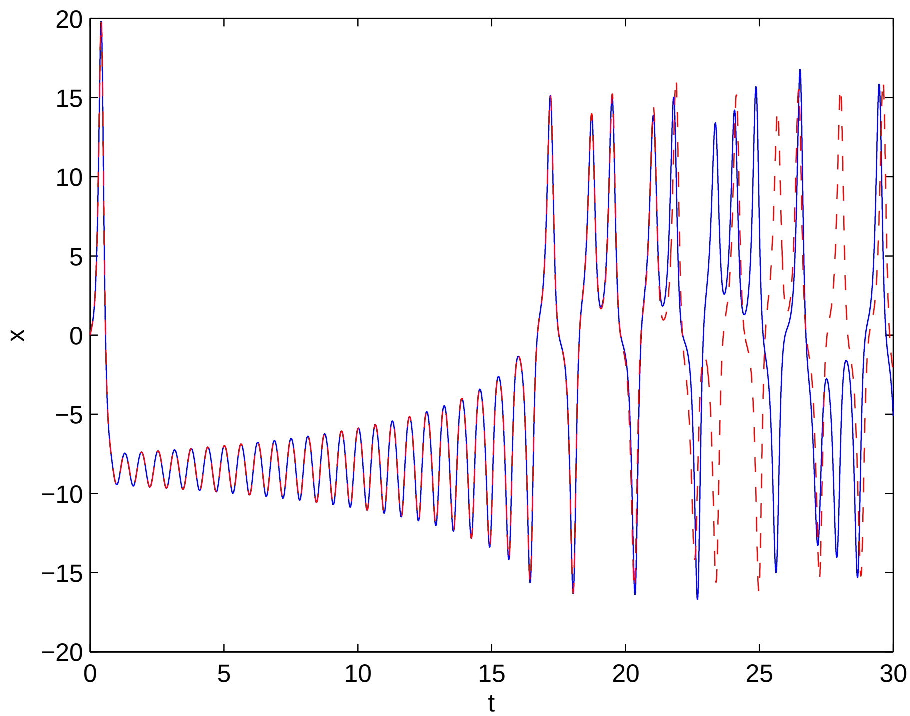

Figure 2: (b) Plot of two solutions for $x(t)$ with slightly differing initial conditions, illustrating the sensitive dependence of solutions on initial conditions.