Problem set 2

Math 207A, Fall 2014

1. Determine the linearized stability of the equilibrium $(x, y, z) = (0, 0, 0)$ of the Lorentz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z, \end{aligned}$$

where $\sigma, r, \beta > 0$. How does the stability change as r increases from 0?

2. Suppose that A is the 3×3 Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(a) Compute e^{tA} from the power series definition. Hint. Write $A = \lambda I + N$ and compute e^{tN}.

(b) Let $\epsilon > 0$. Solve the 3×3 linear system

$$x_t = -\epsilon x + y, \qquad y_t = -\epsilon y + z, \qquad z_t = -\epsilon z \tag{1}$$

subject to the initial condition $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. How do $x(t)$, $y(t)$, $z(t)$ behave as $t \to \infty$? What is the maximum value of $x(t)$ for $0 \le t < \infty$ if $(x_0, y_0, z_0) = (0, 0, 1)$?

(c) Suppose that (1) is the linearization of a 3×3 nonlinear system $\vec{x}_t = \vec{f}(\vec{x})$ at an equilibrium $\vec{x} = 0$. Do you expect (1) to provide a good approximation of solutions of the nonlinear system with initial condition $\vec{x}(0) = \epsilon \vec{y_0}$ when ϵ is small?

3. Write the following 2×2 system for (x, y)

$$\begin{aligned} \dot{x} &= y + \mu x(x^2 + y^2) \\ \dot{y} &= -x + \mu y(x^2 + y^2) \end{aligned}$$

in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. Sketch the phase planes for: (a) $\mu < 0$; (b) $\mu = 0$; (c) $\mu > 0$. Discuss the linear and nonlinear stability of the equilibrium $(x, y) = (0, 0)$. Is the equilibrium hyperbolic?

4. (a) Prove the following version of Gronwall's inequality: If $u(t)$ is a differentiable function on $\mathbb{R}$ such that

$$\dot{u}(t) \le a + bu(t) \quad \text{for all } t \ge 0, \qquad u(0) = u_0,$$

where a, b, u_0 are constants, then

$$u(t) \le u_0 e^{bt} + \frac{a}{b}\left(e^{bt} - 1\right) \quad \text{for all } t \ge 0.$$

(b) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, then the solution $x(t)$ of the IVP

$$x_t = f(x), \qquad x(0) = x_0$$

grows at most exponentially in time and exists for all $-\infty < t < \infty$.

5. The Hénon map on $\mathbb{R}^2$ is defined by

$$\begin{aligned} x_{n+1} &= a - x_n^2 + by_n, \\ y_{n+1} &= x_n, \end{aligned}$$

where a, b are constants. Write a MATLAB script to compute iterates of this map. Plot 10^4 iterates in the orbit with initial condition $x_0 = 1$, $y_0 = 0$ for the parameter values $a = 1.4$, $b = 0.3$. Discuss the long-time behavior of your solution.