Problem set 2
Math 207A, Fall 2014

1. Determine the linearized stability of the equilibrium $(x, y, z) = (0, 0, 0)$ of the Lorentz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z, \end{aligned}$$

where $\sigma, r, \beta > 0$. How does the stability change as r increases from 0?

Solution

- The linearization of the system at $(0, 0, 0)$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_t = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

- The characteristic equation $\det(A - \lambda I) = 0$ for the matrix A in the linearized system is

$$[(\lambda + \sigma)(\lambda + 1) - r\sigma](\lambda + \beta) = 0,$$

  with roots $\lambda = -\beta < 0$ and

$$\lambda = \frac{-(1 + \sigma) \pm \sqrt{(1 + \sigma)^2 + 4\sigma(r - 1)}}{2}.$$

  Both of these roots are real and negative if $0 < r < 1$, since

$$0 < (1 - \sigma)^2 + 4\sigma r = (1 + \sigma)^2 + 4\sigma(r - 1) < (1 + \sigma)^2,$$

  and one root is negative and one positive if $r > 1$.

- It follows that $(0, 0, 0)$ is asymptotically stable if $0 < r < 1$ and unstable if $r > 1$. Since A is singular at $r = 1$, a bifurcation of equilibria can occur as r increases through 1. (In fact, one can show that a supercritical pitchfork bifurcation occurs at $r = 1$.)

2. Suppose that A is the 3×3 Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(a) Compute e^{tA} from the power series definition. HINT. Write $A = \lambda I + N$ and compute e^{tN}.

(b) Let $\epsilon > 0$. Solve the 3×3 linear system

$$x_t = -\epsilon x + y, \qquad y_t = -\epsilon y + z, \qquad z_t = -\epsilon z \tag{1}$$

subject to the initial condition $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. How do $x(t)$, $y(t)$, $z(t)$ behave as $t \to \infty$? What is the maximum value of $x(t)$ for $0 \le t < \infty$ if $(x_0, y_0, z_0) = (0, 0, 1)$?

(c) Suppose that (1) is the linearization of a 3×3 nonlinear system $\vec{x}_t = \vec{f}(\vec{x})$ at an equilibrium $\vec{x} = 0$. Do you expect (1) to provide a good approximation of solutions of the nonlinear system with initial condition $\vec{x}(0) = \epsilon \vec{y_0}$ when ϵ is small?

Solution

- (a) First, note that λI and N commute (the identity matrix commutes with every matrix), so

$$e^{t(\lambda I + N)} = e^{t\lambda I} e^{tN}.$$

- Since $I^n = I$, we have

$$e^{t\lambda I} = \left(1 + t\lambda + \frac{1}{2}t^2\lambda^2 + \cdots + \frac{1}{n!}t^n\lambda^n + \ldots\right) I = e^{\lambda t} I.$$

- The matrix N is nilpotent (meaning that a power of N is zero):

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  and $N^n = 0$ for $n \ge 3$. It follows that

$$\begin{aligned} e^{tN} &= I + tN + \frac{1}{2}t^2 N^2 \\ &= \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

- Combining these results, we get

$$e^{tA} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

- (b) Either back-substitute and solve first-order ODEs for z, y, and x, or use

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{-\epsilon t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

  Then $x(t), y(t), z(t) \to 0$ as $t \to \infty$ since $e^{-\epsilon t}, te^{-\epsilon t}, t^2 e^{-\epsilon t} \to 0$.

- If $x_0 = y_0 = 0$, $z_0 = 1$, then

$$x(t) = \frac{1}{2} t^2 e^{-\epsilon t},$$

  whose maximum value is

$$x\left(\frac{2}{\epsilon}\right) = \frac{2}{e^2 \epsilon^2}.$$

  Thus, even though $x(t) \to 0$ as $t \to \infty$, the solution has large transient behavior when ϵ is small.

- (c) If $z_0 = O(\epsilon)$, then the solution for $x(t)$ grows to $O(\epsilon^{-1})$. We would not expect the linearization of the nonlinear system to remain valid in this circumstance.

Remark. A matrix is said to be normal if it commutes with its transpose (or, for complex matrices, its Hermitian conjugate). Matrices with nontrivial Jordan blocks, like A in this problem, are not normal, and evolution equations with nonnormal matrices (or nonnormal linear operators) may have solutions with large transient growths. This behavior is not apparent from the eigenvalues (or spectrum) of A alone, but one can analyze it further by use of the notion of the pseudo-spectrum of a matrix (or linear operator).

3. Write the following 2×2 system for (x, y)

$$\dot{x} = y + \mu x(x^2 + y^2)$$
$$\dot{y} = -x + \mu y(x^2 + y^2)$$

in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. Sketch the phase planes for: (a) $\mu < 0$; (b) $\mu = 0$; (c) $\mu > 0$. Discuss the linear and nonlinear stability of the equilibrium $(x, y) = (0, 0)$. Is the equilibrium hyperbolic?

Solution

- (a) Writing
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right),$$
we compute that
$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \mu r^3, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = -1.$$

- Thus, if $\mu \neq 0$, then the orbits of the system consist of the equilibrium $(x, y) = (0, 0)$ and counter-clockwise spirals, directed inward toward the origin if $\mu < 0$ and outward toward infinity if $\mu > 0$. If $\mu = 0$, then the orbits consist of the equilibrium $(0, 0)$ and counter-clockwise circles.

- Suppose that $r_0 = r(0) > 0$. It follows from the ODE for r that: (a) if $\mu < 0$, then $r(t) \to 0$ as $t \to \infty$, so $(0, 0)$ is globally asymptotically stable; (b) if $\mu = 0$, then $r(t) = r_0$ is constant, so $(0, 0)$ is stable but not asymptotically stable; (c) if $\mu > 0$, then $r(t) \to \infty$ in finite positive time, so $(0, 0)$ is unstable.

- The linearization of the system at $(0, 0)$ is the $\mu = 0$ system
$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
which has eigenvalues $\lambda = \pm i$. These eigenvalues have zero real part, so the equilibrium is not hyperbolic.

Remark. As this example illustrates, in general, one can't conclude anything about the stability of a nonhyperbolic equilibrium from its linearized stability.

4. (a) Prove the following version of Gronwall's inequality: If $u(t)$ is a differentiable function on $\mathbb{R}$ such that

$$\dot{u}(t) \leq a + bu(t) \quad \text{for all } t \geq 0, \qquad u(0) = u_0,$$

where a, b, u_0 are constants, then

$$u(t) \leq u_0 e^{bt} + \frac{a}{b}\left(e^{bt} - 1\right) \quad \text{for all } t \geq 0.$$

(b) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, then the solution $x(t)$ of the IVP

$$x_t = f(x), \qquad x(0) = x_0$$

grows at most exponentially in time and exists for all $-\infty < t < \infty$.

Solution

- Since $e^{-bt} > 0$, we have

$$\frac{d}{dt}\left[e^{-bt}u\right] = e^{-bt}\left(\dot{u} - bu\right) \leq ae^{-bt}.$$

  It follows that, for $t \geq 0$,

$$\begin{aligned} e^{-bt}u(t) &= u_0 + \int_0^t \frac{d}{ds}\left[e^{-bs}u(s)\right] ds \\ &\leq u_0 + \int_0^t ae^{-bs}\, ds \\ &\leq u_0 + \frac{a}{b}\left[1 - e^{-bt}\right], \end{aligned}$$

  and multiplication of this inequality by e^{bt} gives the result.

- Since f is globally Lipschitz continuous, we have for all $x \in \mathbb{R}^n$ that

$$|f(x)| \leq |f(0)| + |f(x) - f(0)| \leq C + M|x|,$$

  where $C = |f(0)|$ and M is a Lipschitz constant for f.

- Taking the scalar product of the ODE with $2x$, we get that

$$\frac{d}{dt}|x|^2 = 2x \cdot \frac{dx}{dt} = 2x \cdot f(x).$$

It follows from the Cauchy-Schwarz inequality and the global Lipschitz continuity of f that

$$\frac{d}{dt}|x|^2 \leq 2|x|\,|f(x)| \leq 2C|x| + 2M|x|^2 \leq C\left(1 + |x|^2\right) + 2M|x|^2,$$

so Gronwall's inequality gives

$$|x(t)|^2 \leq |x_0|^2 + \frac{C}{B}\left(e^{Bt} - 1\right),$$

for all $t \geq 0$, where $B = C + 2M$. Changing $t \mapsto -t$ and using the same argument, we get the corresponding estimate for $t \leq 0$, with e^{Bt} replaced by $e^{B|t|}$.

- Since the solution remains bounded on every time interval $(-T, T)$, the extension theorem implies that the solution exists for all $-\infty < t < \infty$.

5. The Hénon map on $\mathbb{R}^2$ is defined by

$$x_{n+1} = a - x_n^2 + by_n,$$
$$y_{n+1} = x_n,$$

where a, b are constants. Write a MATLAB script to compute iterates of this map. Plot 10^4 iterates in the orbit with initial condition $x_0 = 1$, $y_0 = 0$ for the parameter values $a = 1.4$, $b = 0.3$. Discuss the long-time behavior of your solution.

Solution

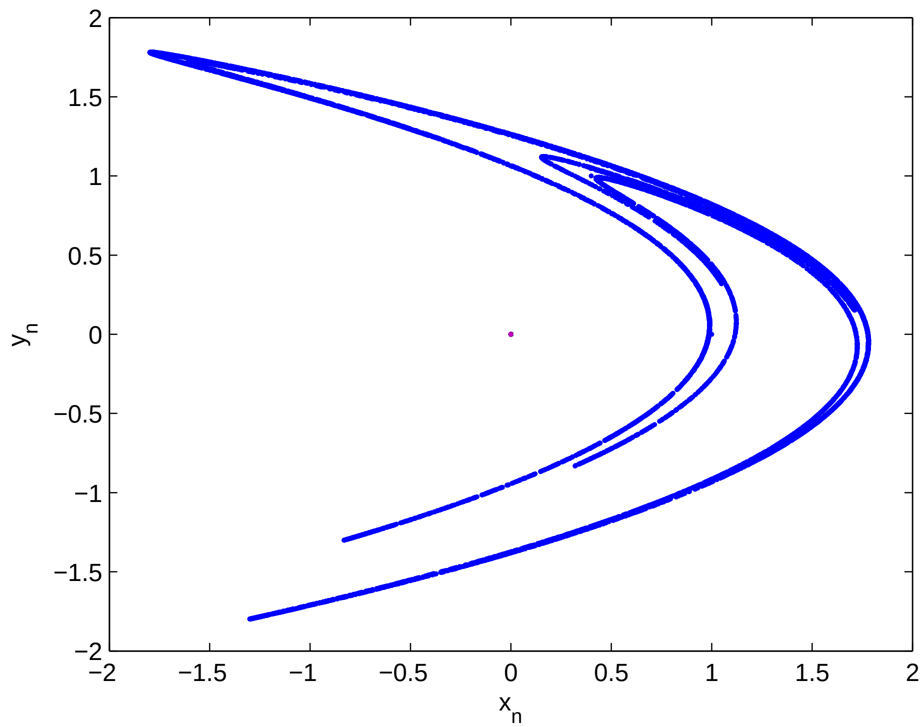

Figure 1: Plot of the first 10,000 iterates of the Hénon map.

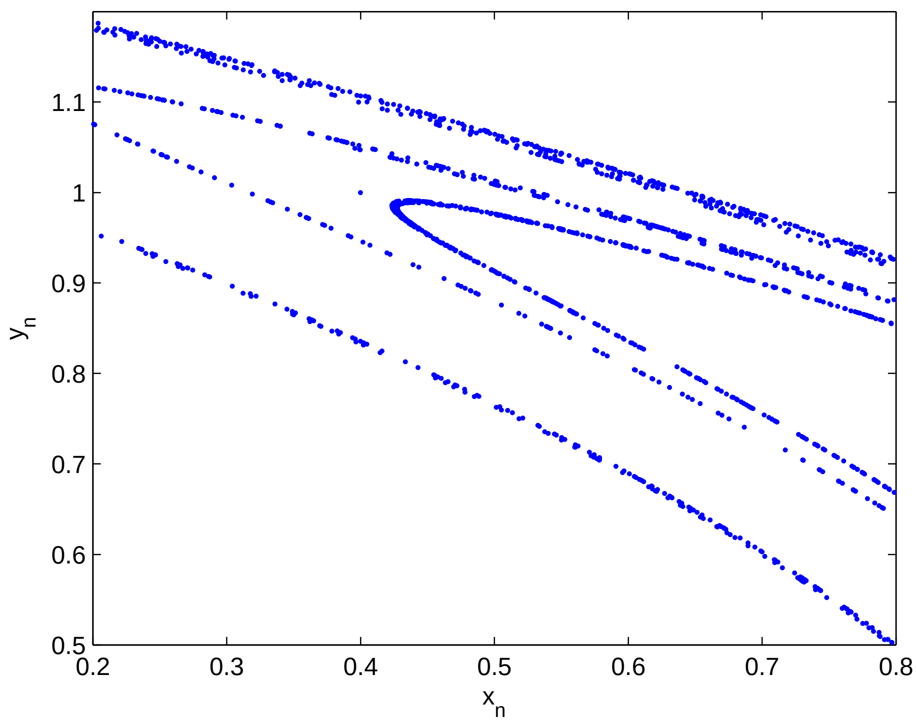

Figure 2: Detail of the Hénon iterates illustrating the transverse Cantor-set structure of its attractor.