# Problem set 3

Math 207A, Fall 2014

**1.** Find the fixed points of the Hénon map

$$x_{n+1} = a - x_n^2 + by_n, \qquad y_{n+1} = x_n,$$

where $a$, $b$ are positive constants, and determine their linearized stability.

**2.** (a) Sketch the phase portraits of the following ODEs on $\mathbb{R}$:

$$\text{(i)} \quad x_t = 1 + x - 2x^3; \qquad \text{(ii)} \quad x_t = \sin(x^2).$$

(b) Sketch the phase portraits of the following ODEs on the circle $\mathbb{T}$:

$$\text{(i)} \quad x_t = 1 - 2\sin x; \qquad \text{(ii)} \quad x_t = 2 - \sin x.$$

**3.** (a) Solve the IVP for the ODE

$$x_t + x = \sin t.$$

Sketch the orbits in $(t, x)$-phase space.

(b) Compute the Poincaré map $P : \mathbb{R} \to \mathbb{R}$ that maps the initial value $x(0)$ to $x(2\pi)$. Indicate some points and their images in your sketch from (a).

(c) Find the fixed point of the Poincaré map and determine its stability. What solution of the original ODE does this fixed point correspond to to?

**4.** (a) Measles is a highly infectious disease with lifelong immunity after recovery. Explain why the following ODEs provide a simple model of a measles epidemic in a population of $N$ individuals with $S(t)$ susceptible members, $I(t)$ infected (and infectious) members, and $R(t)$ recovered (and non-infectious) members:

$$S_t = -\frac{\beta}{N} IS, \qquad I_t = \frac{\beta}{N} IS - \gamma I, \qquad R_t = \gamma I.$$

What are the interpretations of the positive constants $\beta$, $\gamma$? What are their dimensions? Show that these ODEs imply that $S(t) + I(t) + R(t) = N$ remains constant in time.

(b) Suppose that there is no recovery from the disease ($\gamma = 0$ and $R = 0$). Derive a logistic equation for $I(t)$, sketch its phase line, and discuss the long-time behavior of solutions as $t \to \infty$. How would the long-time behavior differ if $\gamma > 0$?

**5.** Consider a scalar autonomous ODE $x_t = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

(a) Prove that a hyperbolic equilibrium $\bar{x} \in \mathbb{R}$ is asymptotically stable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$. HINT. Use a Lyapunov function $V(x) = (x - \bar{x})^2/2$.

(b) In each of the following cases, give an example of such an ODE with a non-hyperbolic equilibrium that is: (i) asymptotically stable; (ii) unstable; (iii) stable but not asymptotically stable.