Problem set 3: Solutions
Math 207A, Fall 2014

1. Find the fixed points of the Hénon map

$$x_{n+1} = a - x_n^2 + by_n, \qquad y_{n+1} = x_n,$$

where a, b are positive constants, and determine their linearized stability.

Solution

- The fixed points $(\bar{x}, \bar{y})$ satisfy

  $$\bar{x} = a - \bar{x}^2 + b\bar{y}, \qquad \bar{y} = \bar{x},$$

  which gives
  $$\bar{x}^2 - (b-1)\bar{x} - a = 0.$$
  This quadratic equation has two real roots for positive a, so there are two fixed points $(\bar{x}, \bar{y}) = (c_\pm, c_\pm)$, where

  $$c_\pm(a, b) = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2},$$

  so that $c_+ > 0$ and $c_- < 0$.

- The linearization of the Hénon map at $(x_n, y_n) = (c, c)$ is

  $$x_{n+1} = -2cx_n + by_n, \qquad y_{n+1} = x_n.$$

  The eigenvalues λ of the Jacobian matrix are given by

  $$\begin{vmatrix} -2c - \lambda & b \\ 1 & -\lambda \end{vmatrix} = 0,$$

  or $\lambda^2 + 2c\lambda - b = 0$. It follows that

  $$\lambda = c \pm \sqrt{c^2 + b}.$$

  These eigenvalues are real since $b > 0$.

- The equilibrium is asymptotically stable if both eigenvalues satisfy $|\lambda| < 1$, and unstable if at least one eigenvalue satisfies $|\lambda| > 1$. For fixed $0 < c < 1/2$, the eigenvalue $\lambda = c + \sqrt{c^2 + b}$ has the largest absolute value, is positive and increases through 1 at $b = 1 - 2c$. For fixed $-1/2 < c < 0$, the eigenvalue $\lambda = c - \sqrt{c^2 + b}$ has the largest absolute value, is negative and decreases through -1 at $b = 1 + 2c$.

- It follows that the equilibrium (c_+, c_+) is stable if $0 < c_+ < 1/2$ and $0 < b < 1 - 2c_+$, otherwise it is unstable, while (c_-, c_-) is stable if $-1/2 < c_- < 0$ and $0 < b < 1 + 2c_-$, otherwise it is unstable.

2. (a) Sketch the phase portraits of the following ODEs on $\mathbb{R}$:

$$\text{(i)} \quad x_t = 1 + x - 2x^3; \qquad \text{(ii)} \quad x_t = \sin(x^2).$$

(b) Sketch the phase portraits of the following ODEs on the circle $\mathbb{T}$:

$$\text{(i)} \quad x_t = 1 - 2\sin x; \qquad \text{(ii)} \quad x_t = 2 - \sin x.$$

Solution

- We omit phase plots and only describe the equilibria and directions of the orbits.

- (a) (i) We have

$$f(x) = 1 + x - 2x^3 = (1 - x)(1 + 2x + 2x^2).$$

  Since $1+2x+2x^2$ has no real roots, there is a single equilibrium $x = 1$. Moreover, $f(x) > 0$ for $x < 1$ and $f(x) < 0$ for $x > 1$.

- (ii) We have $f(x) = \sin(x^2) = 0$ if $x = 0$ or $x = \pm\sqrt{n\pi}$ for $n \in \mathbb{N}$. Moreover $f'(x) = 2x\cos(x^2)$, so

$$f'\left(\pm\sqrt{n\pi}\right) = \pm 2n\pi\cos(n\pi) = \pm 2n\pi(-1)^n.$$

  It follows that $x = 0$ is non-hyperbolic and semi-stable, $x = \sqrt{n\pi}$ is asymptotically stable if n is odd and unstable if n is even, and $x = -\sqrt{n\pi}$ is asymptotically stable if n is even and unstable if n is odd.

- (b) (i) We have $f(x) = 1 - 2\sin x = 0$ if $x = \pi/6, 5\pi/6$. Also, $f(x) < 0$ for $\pi/6 < x < 5\pi/6$, so the corresponding orbit is clockwise, and $f(x) > 0$ for $5\pi/6 < x < 2\pi$, so the corresponding orbit is counter-clockwise. The equilibrium $x = \pi/6$ is asymptotically stable and $x = 5\pi/6$ is unstable.

- (ii) We have $2 - \sin x > 0$ for all $x \in \mathbb{T}$, so the system has no equilibria, and the orbits are clockwise.

3. (a) Solve the IVP for the ODE

$$x_t + x = \sin t.$$

Sketch the orbits in (t, x)-phase space.

(b) Compute the Poincaré map $P : \mathbb{R} \to \mathbb{R}$ that maps the initial value $x(0)$ to $x(2\pi)$. Indicate some points and their images in your sketch from (a).

(c) Find the fixed point of the Poincaré map and determine its stability. What solution of the original ODE does this fixed point correspond to to?

Solution

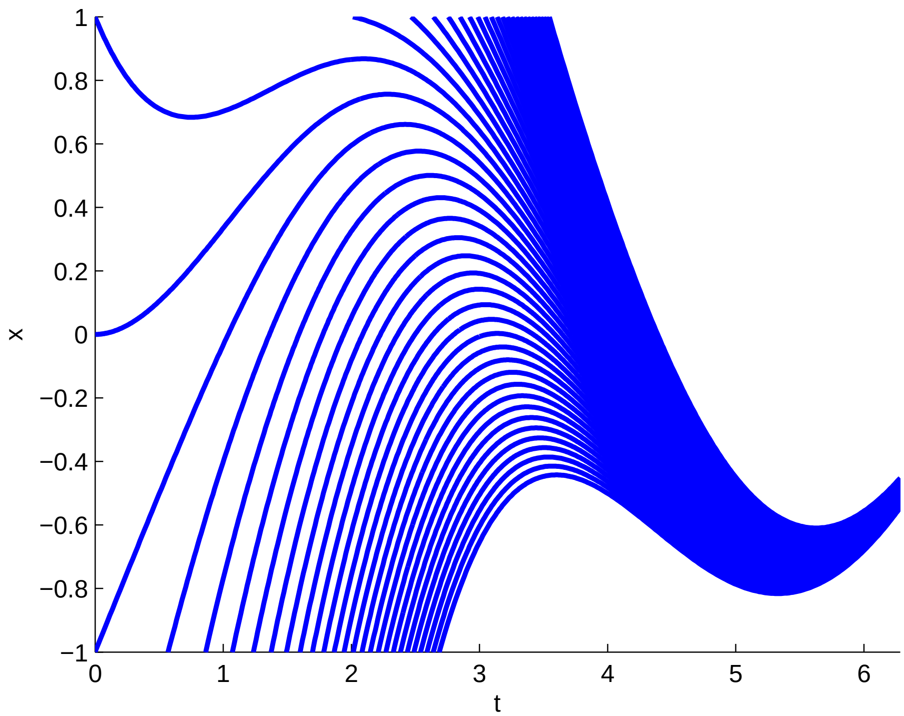

Figure 1: (a) Orbits for $0 \leq t \leq 2\pi$ and $-1 \leq x \leq 1$.

- (b) Multiplying the ODE by the integrating factor e^t and integrating, we get

$$e^t x(t) = x_0 + \int_0^t e^s \sin s \, ds,$$

which gives

$$x(t) = \left(x_0 + \frac{1}{2}\right) e^{-t} + \frac{1}{2} \left(\sin t - \cos t\right).$$

The Poincaré map $x_{n+1} = P(x_n)$ is given by

$$x_{n+1} = e^{-2\pi} \left(x_n + \frac{1}{2}\right) - \frac{1}{2}.$$

For example, $P(0) = (e^{-2\pi} - 1)/2$.

- The fixed point $\bar{x}$ of P satisfies

$$\bar{x} = e^{-2\pi} \left(\bar{x} + \frac{1}{2}\right) - \frac{1}{2},$$

so $\bar{x} = -1/2$. It is asymptotically stable since

$$|P'(\bar{x})| = e^{-2\pi} < 1.$$

- The fixed point corresponds to the 2π-periodic solution of the ODE

$$x(t) = \frac{1}{2} \left(\sin t - \cos t\right).$$

4. (a) Measles is a highly infectious disease with lifelong immunity after recovery. Explain why the following ODEs provide a simple model of a measles epidemic in a population of N individuals with $S(t)$ susceptible members, $I(t)$ infected (and infectious) members, and $R(t)$ recovered (and non-infectious) members:

$$S_t = -\frac{\beta}{N} IS, \qquad I_t = \frac{\beta}{N} IS - \gamma I, \qquad R_t = \gamma I.$$

What are the interpretations of the positive constants β, γ? What are their dimensions? Show that these ODEs imply that $S(t) + I(t) + R(t) = N$ remains constant in time.

(b) Suppose that there is no recovery from the disease ($\gamma = 0$ and $R = 0$). Derive a logistic equation for $I(t)$, sketch its phase line, and discuss the long-time behavior of solutions as $t \to \infty$. How would the long-time behavior differ if $\gamma > 0$?

Solution

- (a) If infected and susceptible individuals encounter each other randomly and uniformly in time, then the probability per unit time of an encounter is proportional to IS. The constant β measures the probability per encounter that the susceptible individual is infected. The constant γ measures the rate at which infected individual recover. Both β and γ have dimensions of $(\text{time})^{-1}$.

- We have $S + I + R = \text{constant}$, since

$$\frac{d}{dt}(S + I + R) = -\frac{\beta}{N} IS + \frac{\beta}{N} IS - \gamma I + \gamma I = 0.$$

- (b) If $\gamma = R = 0$, then $S = N - I$, and

$$I_t = \frac{\beta}{N} I (N - I).$$

  The equilibrium $I = N$ in which all individuals are infected is asymptotically stable, and $I(t) \to N$ as $t \to \infty$ (provided that there are initially some infected individuals so $I(0) > 0$).

- If $\beta, \gamma > 0$, then every individual will eventually get infected and recover, so $S(t), I(t) \to 0$ and $R(t) \to N$ as $t \to \infty$.

5. Consider a scalar autonomous ODE $x_t = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

(a) Prove that a hyperbolic equilibrium $\bar{x} \in \mathbb{R}$ is asymptotically stable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$. Hint. Use a Lyapunov function $V(x) = (x - \bar{x})^2/2$.

(b) In each of the following cases, give an example of such an ODE with a non-hyperbolic equilibrium that is: (i) asymptotically stable; (ii) unstable; (iii) stable but not asymptotically stable.

Solution

- (a) We have
$$\frac{d}{dt}V(x) = (x - \bar{x})\,\dot{x} = (x - \bar{x})\,f(x).$$
Since $f(\bar{x}) = 0$ and f is differentiable at $\bar{x}$, the mean value theorem implies that there exists $\xi = \xi(x)$ between $\bar{x}$ and x such that
$$f(x) = f(x) - f(\bar{x}) = f'(\xi)\,(x - \bar{x})\,,$$
so
$$\frac{d}{dt}V(x) = 2f'(\xi)V(x).$$

- Since $f'(\bar{x}) \neq 0$ and f' is continuous at $\bar{x}$, there exists $\delta > 0$ such that
$$|f'(x) - f'(\bar{x})| \le \frac{1}{2}\,|f'(\bar{x})|$$
for $|x - \bar{x}| \le \delta$.

- First, suppose that $f'(\bar{x}) = -a < 0$, which implies that $f'(x) \le -a/2$ for $|x - \bar{x}| \le \delta$. We claim that if $|x(0) - \bar{x}| \le \delta$, then $|x(t) - \bar{x}| \le \delta$ for all $t \ge 0$. If not, there exists some earliest time $t_0 \ge 0$ after which $|x(t) - \bar{x}|$ exceeds δ. Then $|x(t_0) - \bar{x}| = \delta$ and there is a sequence of times $t_n > t_0$ such that $|x(t_n) - \bar{x}| > \delta$ and $t_n \downarrow t_0$ as $n \to \infty$. But $V(x(t))$, and therefore $|x(t) - \bar{x}|$, is strictly decreasing at $t = t_0$, since
$$\left.\frac{d}{dt}V\left(x(t)\right)\right|_{t=t_0} \le -\frac{1}{2}a\delta^2 < 0,$$
so there exists $\eta > 0$ such that $|x(t) - \bar{x}| < \delta$ for $t_0 < t < t_0 + \eta$, which is a contradiction.

- It follows that if $|x(0) - \bar{x}| \leq \delta$, then $f'(\xi) < -a/2$ and
$$\frac{d}{dt} V(x) \leq -aV(x)$$
for all $t \geq 0$. Gronwall's inequality implies that
$$V(x(t)) \leq V(x(0))e^{-at}$$
for all $t \geq 0$, which shows that $\bar{x}$ is asymptotically stable. Explicitly, given $\epsilon > 0$, we have $|x(t) - \bar{x}| < \epsilon$ for all $t \geq 0$ whenever $|x(0) - \bar{x}| < \eta$ where $\eta = \min(\delta, \epsilon)$, so $\bar{x}$ is stable, and $x(t) \to \bar{x}$ as $t \to \infty$ if $|x(0) - \bar{x}| < \delta$, since $V(x(t)) \to 0$ as $t \to \infty$, so $\bar{x}$ is asymptotically stable.

- On the other hand, if $f'(\bar{x}) = a > 0$, then a similar argument backward in time ($t \mapsto -t$) shows that if $|x(0) - \bar{x}| < \delta$, then
$$V(x(t)) \leq V(x(0))e^{at}$$
for all $t \leq 0$ provided that $|x(0) - \bar{x}| < \delta$. Applying this result to the solution $x(t + T)$, where $T \geq 0$ is an arbitrary time shift, we have
$$V(x(t + T)) \leq V(x(T))e^{at},$$
for all $t \leq 0$, provided that $|x(T) - \bar{x}| < \delta$, and setting $t = -T$, we get
$$V(x(0))\, e^{aT} \leq V(x(T)).$$
This result shows that $|x(t) - \bar{x}| > \delta/2$ for sufficiently large $t > 0$ however close to $\bar{x}$ we choose $x(0) \neq \bar{x}$, meaning that $\bar{x}$ is not stable.

- (b) Examples with a non-hyperbolic equilibrium at $x = 0$ are:
$$\begin{aligned} &\text{(i)} \quad x_t = -x^3 \qquad \text{(asymptotically stable)};\\ &\text{(ii)} \quad x_t = x^3 \qquad \text{(unstable)};\\ &\text{(iii)} \quad x_t = 0 \qquad \text{(stable but not asymptotically stable)}. \end{aligned}$$
A more interesting example for (iii) is $x_t = f(x)$ where
$$f(x) = \begin{cases} x^3 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
This has infinitely many hyperbolic equilibria at $x = 1/(n\pi)$, for $n \in \mathbb{Z}$, which alternate between asymptotically stable and unstable and accumulate at the stable but not asymptotically stable non-hyperbolic equilibrium $x = 0$.