

PROBLEM SET 4
Math 207A, Fall 2014

1. Sketch the bifurcation diagram and phase lines for the ODE

$$x_t = \mu x + x^3 - x^5,$$

and classify the bifurcations that occur. What would happen if the parameter μ is slowly increased from $-\infty$ to ∞ , and then decreased back to $-\infty$? Where will the symmetry of the system under reflections $x \mapsto -x$ play a role?

Solution

- The equilibria satisfy

$$x(\mu + x^2 - x^4) = 0,$$

so $x = 0$ or

$$x^2 = \frac{1 \pm \sqrt{1 + 4\mu}}{2}.$$

These roots are both complex if $\mu < -1/4$, both real and positive if $-1/4 < \mu < 0$, and one is real and positive the other real and negative if $\mu > 0$. It follows that there are no equilibria in addition to $x = 0$ if $\mu < -1/4$, four additional equilibria if $-1/4 < \mu < 0$, and two additional equilibria if $\mu > 0$.

- A subcritical pitchfork bifurcation occurs at $(x, \mu) = (0, 0)$, and supercritical saddle-node bifurcations occur at $(x, \mu) = \pm(1/2, -1/4)$.
- We have $f_x(0, \mu) = \mu$, so the equilibrium $x = 0$ is asymptotically stable for $\mu < 0$ and unstable if $\mu > 0$. Similarly, one finds that the branches

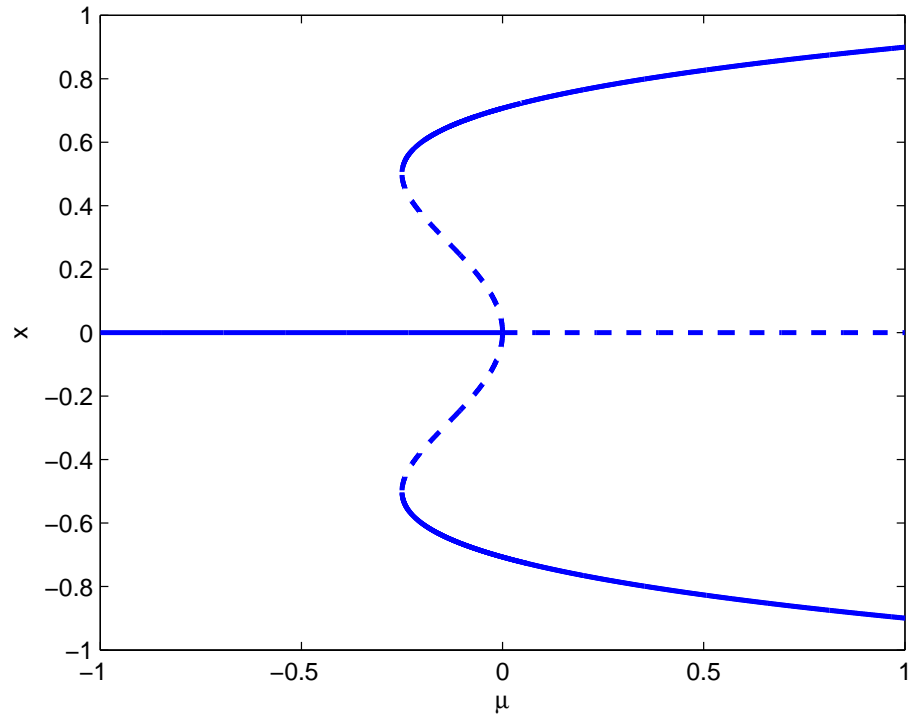
$$x = \pm \sqrt{\frac{1 + \sqrt{1 + 4\mu}}{2}}$$

are asymptotically stable if $\mu > -1/4$, and the branches

$$x = \pm \sqrt{\frac{1 - \sqrt{1 + 4\mu}}{2}}$$

are unstable if $-1/4 < \mu < 0$.

- For $\mu < -1/4$, the equilibrium $x = 0$ is globally asymptotically stable. The system will remain in this state as μ is increased quasistatically to 0. As μ is increased through 0, the system jumps to one of the stable equilibrium branches near $x = \pm 1$. Which one of these symmetric states occurs is not determined by the system, but will depend on external asymmetries, e.g., in the initial data, noise, or imperfections in the system. As μ is increased further, the solution remains on the same stable equilibrium branch.
- As μ is decreased quasistatically from large values, the solution retraces the previous stable equilibrium branch until $\mu = -1/4$ (hysteresis) and then jumps to $x = 0$ for $\mu < -1/4$. The bifurcation diagram is below.



2. Sketch the bifurcation diagrams and phase lines for x versus μ

$$x_t = a + \mu x - x^3$$

for: (i) $a < 0$; (ii) $a = 0$; (iii) $a > 0$. Classify the bifurcations that occur.

Solution

- The equilibria satisfy $x^3 - \mu x - a = 0$. The discriminant of this cubic is $\Delta = 4\mu^3 - 27a^2$. The cubic has three real roots if $\Delta > 0$ and one real root if $\Delta < 0$. (To see this, consider the values of the cubic at the points $3x^2 - \mu = 0$ where it attains a local maximum and minimum for $\mu > 0$.)
- If $a = 0$, then we have a supercritical pitchfork bifurcation at $(x, \mu) = (0, 0)$: there is one asymptotically stable solution $x = 0$ for $\mu < 0$; and three solutions $x = 0$ (unstable), $x = \pm\sqrt{\mu}$ (asymptotically stable) for $\mu > 0$.
- If $a \neq 0$ then there is one solution if $\mu < \mu_0$ and three solutions if $\mu > \mu_0$ where

$$\mu_0 = 3 \left(\frac{a}{2}\right)^{2/3}.$$

A supercritical saddle-node bifurcation occurs at the point (x_0, μ_0) , where

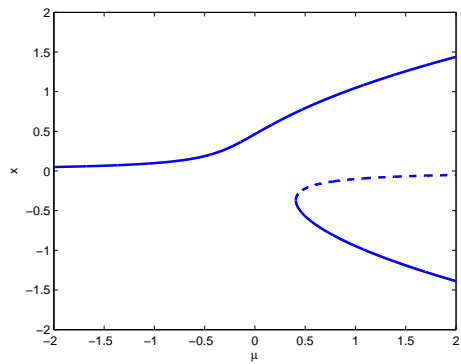
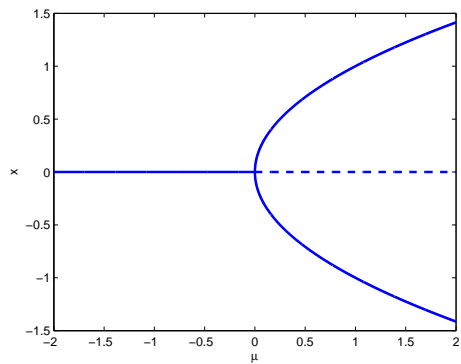
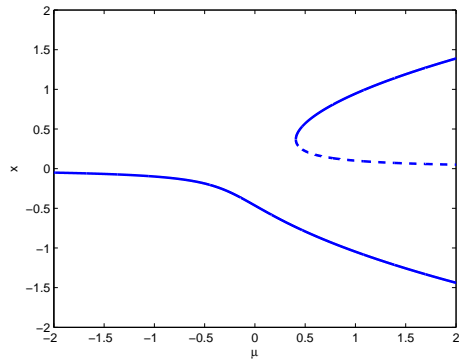
$$x_0 = -\left(\frac{a}{2}\right)^{1/3}$$

and $f(x, \mu) = a + \mu x - x^3$ satisfies

$$\begin{aligned} f(x_0, \mu_0) &= a + \mu_0 x_0 - x_0^3 = 0, \\ \frac{\partial f}{\partial x}(x_0, \mu_0) &= \mu_0 - 3x_0^2 = 0, \\ \frac{\partial f}{\partial \mu}(x_0, \mu_0) &= x_0 \neq 0, \\ \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) &= -6x_0 \neq 0. \end{aligned}$$

Remark. The constant term $a \neq 0$ breaks the reflectional symmetry $x \mapsto -x$ of the pitchfork bifurcation, which splits into a single asymptotically stable

branch with no bifurcations, and two branches (one asymptotically stable, the other unstable) that originate at a supercritical saddle-node bifurcation. Here are the bifurcation diagrams for $a = -0.1$, $a = 0$, and $a = 0.1$, respectively:



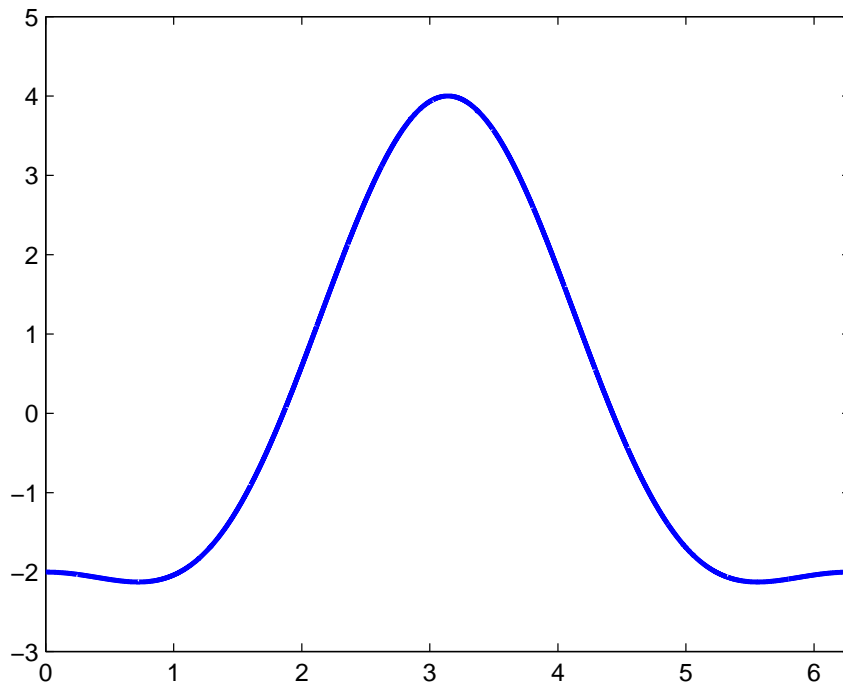
3. Sketch the bifurcation diagram and phase circles for the periodic ODE

$$x_t = -\mu + 2 + \cos 2x - 3 \cos x,$$

and classify the bifurcations that occur. (See Exercise 2.14 in the text for more help.)

Solution

- Graphically, one can read off the equilibria from the intersections of the horizontal line $y = \mu - 2$ with the graph $y = \cos 2x - 3 \cos x$ (see the figure for the graph $y = \cos 2x - 3 \cos x$).



- Writing $f(x, \mu) = -\mu + 2 + \cos 2x - 3 \cos x$, the possible bifurcations of equilibria occur when $f(x, \mu) = 0$ and $f_x(x, \mu) = 0$, which implies that

$$\mu = 2 + \cos 2x - 3 \cos x, \quad 2 \sin 2x = 3 \sin x.$$

Since $\sin 2x = 2 \sin x \cos x$, either $\sin x = 0$, so $x = 0, \pi$, or $\cos x = 3/4$.

- There are no equilibria for $\mu < -1/8$ or $\mu > 6$.
- If $x = \cos^{-1}(3/4)$, or $x \approx 0.7227, 5.5605$, then $\mu = -1/8$ and two supercritical-saddle-node bifurcations occur at this value of μ . There are four equilibria for $-1/8 < \mu < 0$. By looking at the sign of $f(x, \mu)$, one can see that they are stable, unstable, stable, and unstable in order of increasing $0 < x < 2\pi$.
- At $(x, \mu) = (0, 0) = (2\pi, 0)$ there is a subcritical saddle-node bifurcation, and two of these equilibria annihilate each other. There are two equilibria for $0 < \mu < 6$. Finally at $(x, \mu) = (\pi, 6)$, there is a second subcritical saddle-node bifurcation, and the remaining two equilibria annihilate each other.

4. The following PDE for $u(x, t)$, called Burgers equation, is a simply model of the Navier-Stokes equations for viscous fluids

$$u_t + uu_x = u_{xx}$$

(a) Look for traveling wave solutions of the form $u = f(x - ct)$, and derive a first-order ODE for f .

(b) Show that the PDE has traveling wave solutions with $u \rightarrow u_-$ as $x \rightarrow -\infty$ and $u \rightarrow u_+$ as $x \rightarrow \infty$, where u_{\pm} are constants, if $u_- \geq u_+$ and the wave-velocity c is given by

$$c = \frac{u_+ + u_-}{2}.$$

Solve for the traveling wave solution in that case.

Solution

- (a) Using $u = f(x - ct)$ in the PDE, we get that

$$-cf' + ff' = f''.$$

We can integrate this ODE once to get

$$f' = \frac{1}{2}f^2 - cf + b,$$

where b is a constant of integration.

- (b) This ODE has two equilibria $f = u_+ < u_-$, say, if

$$\frac{1}{2}f^2 - cf + b = \frac{1}{2}(f - u_-)(f - u_+),$$

which requires that $c = (u_+ + u_-)/2$. The ODE has a decreasing solution $f(x)$ such that $f(x) \rightarrow u_-$ as $x \rightarrow -\infty$ and $f(x) \rightarrow u_+$ as $x \rightarrow \infty$, which is the required traveling wave solution. (Draw the phase line.)

- Solving the PDE by separating variables and using partial fractions to compute the f -integral, we find that the corresponding traveling wave solution for $u_+ < u < u_-$ can be written as

$$u(x, t) = \frac{1}{2}(u_- + u_+) - \frac{1}{2}(u_- - u_+) \tanh \left[\frac{1}{4}(u_- - u_+)(x - ct) \right],$$

where

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

Remark. Burgers equation provides a good approximation for unidirectional, small-amplitude nonlinear sound waves in compressible fluid mechanics. This traveling wave solution describes the viscous profile of a weak planar shock wave. In this approximation, the shock speed $c = (u_+ + u_-)/2$ is the average of the states on either side of the shock, and the fact that $u_- > u_+$ corresponds to the condition that the entropy of the fluid increases across a shock.