PROBLEM SET 5: SOLUTIONS
Math 207A, Fall 2014

0. Sketch "staircase pictures" in the (x, y) plane with graphs $y = ax$ and $y = x$ for one or two orbits of the linear scalar map

$$x_{n+1} = ax_n$$

in the following cases: (a) $a < -1$; (b) $a = -1$; (c) $-1 < a < 0$; (d) $a = 0$; (e) $0 < a < 1$; (f) $a = 1$; (g) $a > 1$.

Solution

- See Figure 3.3, p. 74 in the text by Hale and Koçak.

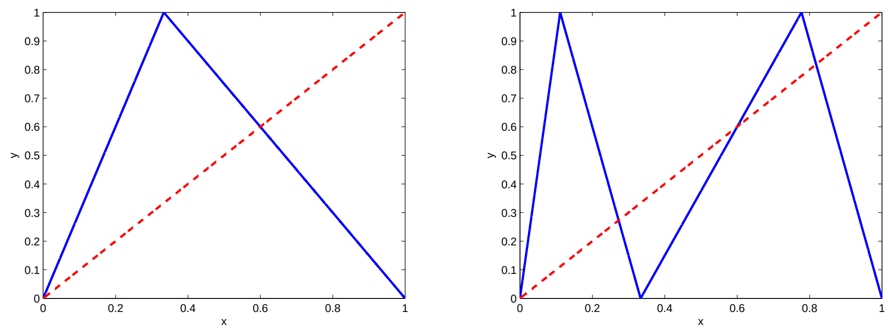

Figure 1: Graphs of $y = f(x)$ (left) and $y = f^2(x)$ (right). The dotted red line is $y = x$; its intersections with the graphs correspond to fixed points of the maps.

1. Define $f : [0, 1] \to [0, 1]$ by

$$f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1/3 \\ \frac{3}{2}(1 - x) & \text{if } 1/3 < x \le 1 \end{cases}$$

and consider the discrete dynamical system

$$x_{n+1} = f(x_n).$$

(a) Find the fixed points of f in $[0, 1]$ and determine their stability.

(b) Find the period two orbit and determine its stability.

Solution

- (a) If $\bar{x} = f(\bar{x})$, then either: (i) $0 \le \bar{x} \le 1/3$ and $\bar{x} = 3\bar{x}$, which gives $\bar{x} = 0$; or (ii) $1/3 < \bar{x} \le 1$ and $\bar{x} = (3/2)(1 - \bar{x})$, which gives $\bar{x} = 3/5$ (see Figure 1).

- We have $f'(0) = 3 > 1$, so $\bar{x} = 0$ is unstable, and

$$f'\left(\frac{3}{5}\right) = -\frac{3}{2} < -1,$$

so $\bar{x} = 3/5$ is also unstable.

- (b) A period two orbit is a fixed point of f^2, which is given by (see Figure 1)

$$f^2(x) = \begin{cases} 9x & \text{if } 0 \le x \le 1/9, \\ \frac{3}{2}(1-3x) & \text{if } 1/9 < x \le 1/3, \\ \frac{3}{2}(1-\frac{3}{2}(1-x)) & \text{if } 1/3 \le x \le 7/9, \\ \frac{9}{2}(1-x) & \text{if } 7/9 < x \le 1. \end{cases}$$

  In addition to the fixed points of f, the iterated function f^2 has fixed points at $x = 3/11$, $x = 9/11$. These correspond to a minimal-period 2 orbit of f: $f(3/11) = 9/11$, $f(9/11) = 3/11$.

- We have

$$(f^2)'\left(\frac{3}{11}\right) = (f^2)'\left(\frac{9}{11}\right) = -\frac{9}{2},$$

  whose absolute value is greater than 1, so the period-2 orbit is unstable.

Remark. In studying problems that involve some class of functions (here iterated maps), it's often useful to consider piecewise linear functions; they may have similar qualitative behavior to more general functions but be easier to analyze.

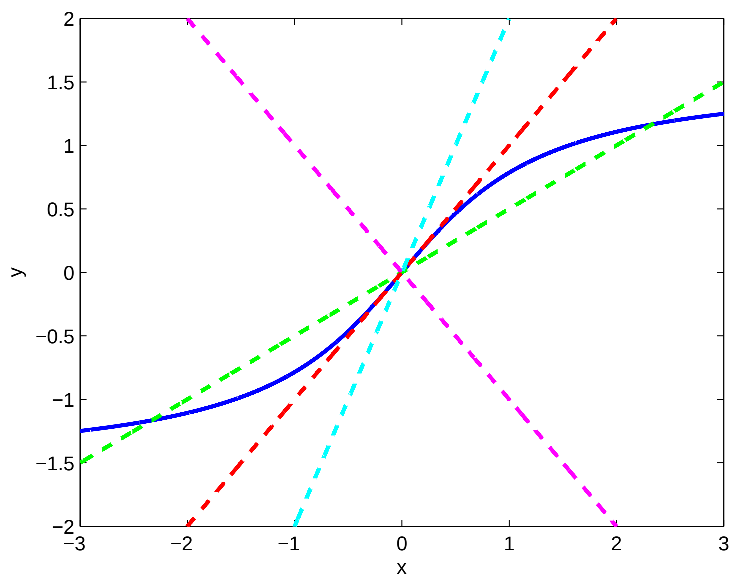

Figure 2: Graphs of $y = \tan^{-1} x$ (blue) and $y = -2x/\mu$ for: $\mu = -4$ (green); $\mu = -2$ (red); $\mu = -1$ (cyan); $\mu = 2$ (magenta).

2. Find the fixed points of the system

$$x_{n+1} = -\frac{\mu}{2} \tan^{-1} x_n$$

and determine their stability. Show that a period-doubling bifurcation occurs at $\mu = 2$. Is the resulting period-two orbit stable or unstable?

Solution

- The fixed points of the map

  $$f(x; \mu) = -\frac{\mu}{2} \tan^{-1} x$$

  correspond to intersections of the line $\mu y = -2x$ with the graph $y = \tan^{-1} x$. (See Figure 2.)

- If $\mu < -2$, then the slope of the line is positive and less than the slope of the inverse tangent at $x = 0$, and there are three such points with

  $$x = 0, \quad x = \pm \bar{x}(\mu)$$

where $\bar{x}(\mu) > 0$ satisfies

$$-\frac{2}{\mu}\bar{x} = \tan^{-1}\bar{x}.$$

If $\mu \geq -2$, then there is a unique fixed point at $x = 0$.

- If $\mu = -2$, then the line is tangent to the graph of the inverse tangent at $x = 0$, and a subcritical pitchfork bifurcation occurs as μ increases through -2. There are no other bifurcations of fixed points.

- We have
$$f_x(x;\mu) = -\frac{\mu}{2}\frac{1}{1+x^2}.$$
Thus, $f_x(0;\mu) = -\mu/2$, so $|f_x(0,\mu)| < 1$ if $|\mu| < 2$ and $|f_x(0,\mu)| > 1$ if $|\mu| > 2$. It follows that the fixed point $x = 0$ is asymptotically stable if $|\mu| < 2$ and unstable if $|\mu| > 2$.

- The fixed point $x = 0$ gains stability as μ increases through -2 and the eigenvalue $f_x(0;\mu)$ decreases through 1 (corresponding to a bifurcation of fixed points); and loses stability as μ increases through 2 and the eigenvalue $f_x(0;\mu)$ decreases through -1 (corresponding to a period doubling bifurcation).

- If $\mu < -2$, then
$$f_x\left(\pm\bar{x}(\mu);\mu\right) = -\frac{\mu}{2}\frac{1}{1+\bar{x}^2(\mu)}.$$
The graph of $y = \tan x$ has smaller slope than the line $y = -2x/\mu$ at $x = \bar{x}(\mu)$, so $0 < f_x\left(\bar{x}(\mu);\mu\right) < 1$, and these fixed points are stable. This claim is clear geometrically, but we omit an analytical proof.

3. Consider the discrete dynamical system on the circle for $x_n \in \mathbb{T}$

$$x_{n+1} = x_n + \mu \qquad (\text{mod } 2\pi)$$

corresponding to rotation by an angle $\mu \in \mathbb{T}$. Describe the structure of the orbits and how they depend on μ.

Solution

- If $\mu = 0$, then every point is a fixed point.
- If μ is nonzero and a rational multiple of 2π, so that

  $$\mu = \frac{2\pi p}{q}$$

  where $p, q \in \mathbb{Z}$ are nonzero, relatively prime integers, then every orbit is periodic with minimal period q.
- If $\mu = 2\pi x$ where $x \in \mathbb{R}\setminus\mathbb{Q}$ is irrational, then every orbit is nonperiodic and dense in $\mathbb{T}$. In fact, according to Weyl's equidistribution theorem, the points $\{x_n : n \in \mathbb{N}\}$ in an orbit are uniformly distributed on $\mathbb{T}$. That is, if $I \subset \mathbb{T}$ is any interval, then

  $$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : x_n \in I\}}{N} = \frac{|I|}{2\pi},$$

  where $|I|$ denotes the length of I and $\#S$ denotes the number of points in $S \subset \mathbb{T}$.

4. Write a MATLAB script to solve the system

$$x_{n+1} = (1-\mu)x_n + \mu x_n^3,$$

and investigate the dynamics of solutions. In particular, investigate numerically the sequence of period-doubling bifurcations that accumulates at $\mu \approx 3.5980\ldots$.

Solution

- For $\mu \neq 0$, the map

  $$f(x;\mu) = (1-\mu)x + \mu x^3$$

  has three fixed points $x = 0$ and $x = \pm 1$, which are independent of μ.

- We have

  $$f_x(x,\mu) = 1 - \mu + 3\mu x^2.$$

- It follows that $f_x(0;\mu) = 1-\mu$, so $|f_x(0;\mu)| < 1$ and $x = 0$ is asymptotically stable for $0 < \mu < 2$ and unstable if $\mu < 0$ or $\mu > 2$. There is a possible bifurcation of fixed points at $(x,\mu) = (0,0)$, where $f_x(0;\mu)$ decreases through 1, and a possible period doubling bifurcation at $(x,\mu) = (0,2)$, where $f_x(0;\mu)$ decreases through -1.

- Similarly, $f_x(\pm 1;\mu) = 1 + 2\mu$, so $x = \pm 1$ is asymptotically stable for $-1/2 < \mu < 0$ and unstable if $\mu < -1/2$ or $\mu > 0$. There is a possible bifurcation of fixed points at $(x,\mu) = (\pm 1, 0)$, where $f_x(\pm 1;\mu)$ increases through 1, and a possible period doubling bifurcation at $(x,\mu) = (\pm 1, -1/2)$, where $f_x(\pm 1;\mu)$ increases through -1.

- The period-doubling bifurcations occur between the fixed points $x = \pm 1$. I posted a .avi file, made by Robert Bassett, on the class SmartSite which shows the dynamics of the first hundred iterates of the map as μ increases from 1 to 4.