Problem set 6

Math 207A, Fall 2014

1. Sketch phase planes of the following 2×2 linear systems:

$$\text{(a)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 4 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(b)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(c)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(d)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(e)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 2 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(f)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In each case, classify the equilibrium $(x, y) = (0, 0)$ (as a saddle point, node etc.), determine its stability, and say if it is hyperbolic or non-hyperbolic.

2. Two $n \times n$ linear systems $\vec{x}_t = A\vec{x}$, $\vec{y}_t = B\vec{y}$ are said to be differentiably equivalent if there is a diffeomorphism (i.e., a differentiable map with differentiable inverse) $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $\vec{y}(t) = h(\vec{x}(t))$ is a solution of $\vec{y}_t = B\vec{y}$ if and only if $\vec{x}(t)$ is a solution of $\vec{x}_t = A\vec{x}$. Show that if $\vec{x}_t = A\vec{x}$ and $\vec{y}_t = B\vec{y}$ are differentiably equivalent, then A and B have the same eigenvalues. Is differentiable equivalence a useful way to classify the qualitative behavior of linear systems? Explain your answer.

3. Consider the following 2×2 system of ODEs

$$x_t = x - y, \qquad y_t = x + y - 2xy.$$

(a) Find the equilibria.

(b) Linearize the system around the equilibria and classify them.

(c) Sketch the phase plane of the system.

(d) Discuss the asymptotic behavior of solutions as $t \to \infty$. Indicate different regions of the phase plane that correspond to different types of asymptotic behavior.