Problem set 6: Solutions

Math 207A, Fall 2014

1. Sketch phase planes of the following 2×2 linear systems:

$$\text{(a)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 4 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(b)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(c)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(d)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(e)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 2 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\text{(f)} \quad \begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In each case, classify the equilibrium $(x, y) = (0, 0)$ (as a saddle point, node etc.), determine its stability, and say if it is hyperbolic or non-hyperbolic.

Solution

- Phase planes are shown in Figure 1. They are drawn using `pplane8.m`, which is available at `http://math.rice.edu/~dfield/`.

- (a) The eigenvalues and eigenvectors are

  $$\lambda = \pm 6i, \qquad \vec{r} = \begin{pmatrix} 2 \\ \pm 3i \end{pmatrix}.$$

  The origin is a center (stable but not asymptotically stable, and non-hyperbolic).

- (b) The eigenvalues and eigenvectors are

  $$\lambda = \pm 6, \qquad \vec{r} = \begin{pmatrix} 2 \\ \pm 3 \end{pmatrix}.$$

  The origin is a saddle point (unstable and hyperbolic).

- (c) The eigenvalues and eigenvectors are

$$\lambda = 2, \qquad \vec{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1, and the matrix is not diagonalizable. The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

The origin is an unstable node (hyperbolic).

- (d) The eigenvalues and eigenvectors are

$$\lambda_1 = 0, \quad \lambda_2 = 4, \qquad \vec{r}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The matrix is singular, and there is a line of equilibria at

$$(x, y) = (c, 2c).$$

The origin is not an isolated equilibrium (unstable and non-hyperbolic).

- (e) The eigenvalues and eigenvectors are

$$\lambda = 1 \pm 3i, \qquad \vec{r} = \begin{pmatrix} 2 \\ 1 \pm 3i \end{pmatrix}.$$

The origin is an unstable spiral point (hyperbolic).

- (f) The eigenvalues and eigenvectors are

$$\lambda_1 = -2, \quad \lambda_2 = -1, \qquad \vec{r}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The origin is a stable node (asymptotically stable and hyperbolic).

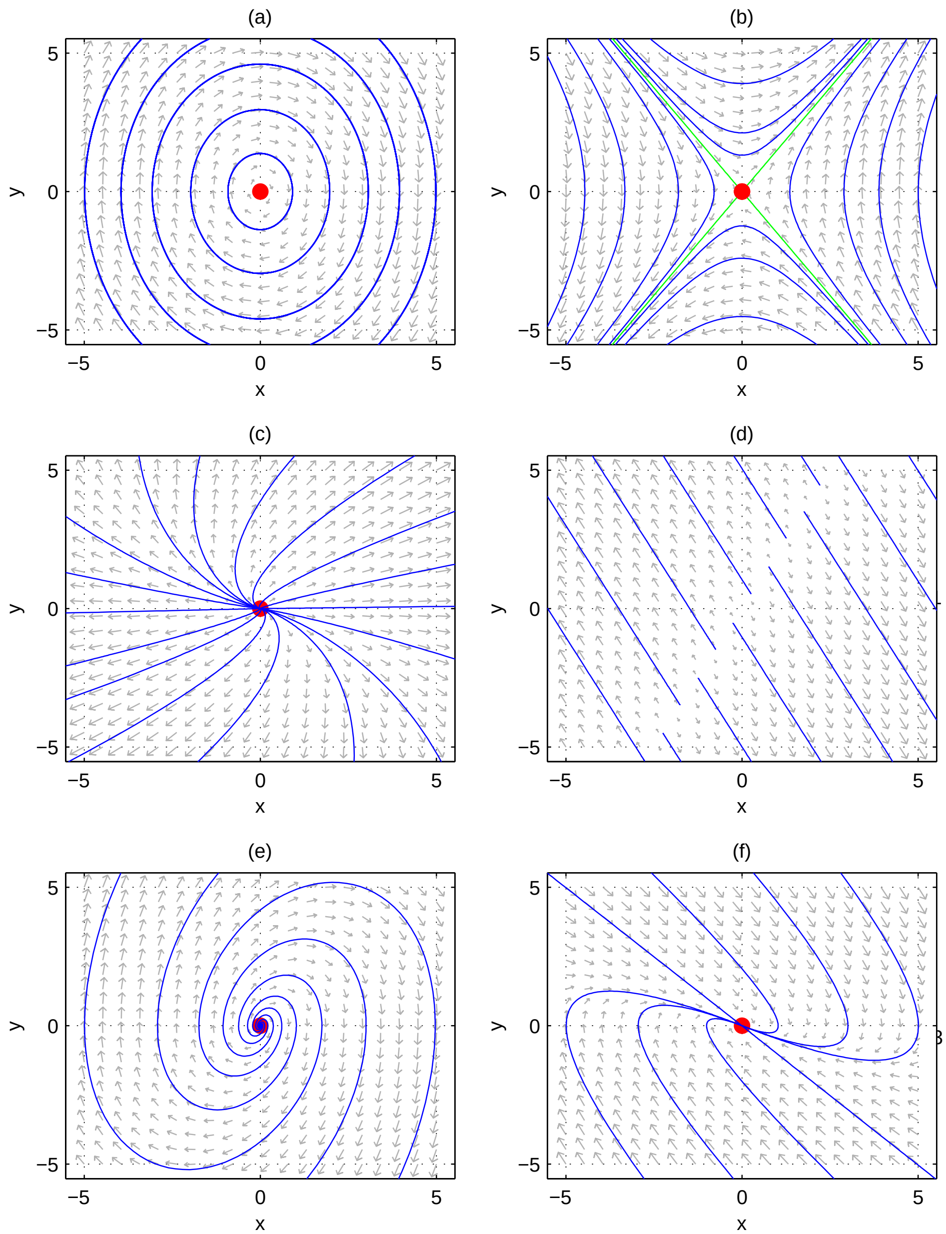

Figure 1: Phase planes for 1(a)–1(f).

2. Two $n \times n$ linear systems $\vec{x}_t = A\vec{x}$, $\vec{y}_t = B\vec{y}$ are said to be differentiably equivalent if there is a diffeomorphism (i.e., a differentiable map with differentiable inverse) $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $\vec{y}(t) = h(\vec{x}(t))$ is a solution of $\vec{y}_t = B\vec{y}$ if and only if $\vec{x}(t)$ is a solution of $\vec{x}_t = A\vec{x}$. Show that if $\vec{x}_t = A\vec{x}$ and $\vec{y}_t = B\vec{y}$ are differentiably equivalent, then A and B have the same eigenvalues. Is differentiable equivalence a useful way to classify the qualitative behavior of linear systems? Explain your answer.

Solution

- Consider, more generally, two $n \times n$ nonlinear systems

$$x_t = f(x), \qquad y_t = g(y).$$

  Suppose that the change of variable $y = h(x)$ maps the first system into the second. (We don't indicate vectors explicitly.)

- Using the chain rule and the x-equation, we get that

$$y_t = Dh(x) \cdot x_t = Dh(x) \cdot f(x),$$

  where the linear map $Dh(x) : \mathbb{R}^n \to \mathbb{R}^n$ is the derivative of $h : \mathbb{R}^n \to \mathbb{R}^n$ at x. (The matrix of Dh is the Jacobian matrix of h.) If ODEs have the same solutions, then they have the same vector fields, so it follows that

$$g\left(h(x)\right) = Dh(x) \cdot f(x). \tag{1}$$

- Now suppose that $\bar{x}$ is an equilbrium solution for the x-equation, meaning that $f(\bar{x}) = 0$, and let $\bar{y} = h(\bar{x})$ be the corresponding equilbrium of the y-equation. The linearizations of the x, y-equations at $x = \bar{x}$, $y = \bar{y}$ are:

$$x_t = Ax, \quad A = Df(\bar{x}); \qquad y_t = By, \quad B = Dg(\bar{y}).$$

- Differentiating (1) with respect to x and using the chain rule, we get that

$$Dg\left(h(x)\right) \cdot Dh(x) = D^2h(x) \cdot f(x) + Dh(x) \cdot Df(x); \tag{2}$$

  or, in component notation with $y_i = h_i(x_j)$,

$$g_i = \frac{\partial y_i}{\partial x_j} f_j, \qquad \frac{\partial g_i}{\partial y_j}\frac{\partial y_j}{\partial x_k} = \frac{\partial^2 y_i}{\partial x_j \partial x_k} f_j + \frac{\partial y_i}{\partial x_j}\frac{\partial f_j}{\partial x_k},$$

  where we use the summation convention over repeated j-indices.

- Setting $x = \bar{x}$ in (2) and using the fact that $f(\bar{x}) = 0$, we get that

  $$B \cdot Dh(\bar{x}) = Dh(\bar{x}) \cdot A.$$

  Since h is a diffeomorphism, $Dh(\bar{x})$ is nonsingular, and

  $$B = Dh(\bar{x}) \cdot A \cdot Dh(\bar{x})^{-1}.$$

  This means that the matrices of A, B are similar; in particular, they have the same eigenvalues.

- Differentiable equivalence is too fine a notion to provide a useful classification of the qualitative dynamics of ODEs. For example, we would like to regard any two planar linear systems for which the origin is a saddle point as equivalent. However, if one equilibrium has eigenvalues $\lambda = -1, 1$ and another has eigenvalues $\lambda = -1.001, 1$, then the corresponding systems are not differentiably equivalent. This fact explains why we use the weaker notion of topological equivalence to classify dynamical systems, even though it is harder to work with.

3. Consider the following 2×2 system of ODEs

$$x_t = x - y, \qquad y_t = x + y - 2xy. \tag{3}$$

(a) Find the equilibria.
(b) Linearize the system around the equilibria and classify them.
(c) Sketch the phase plane of the system.
(d) Discuss the asymptotic behavior of solutions as $t \to \infty$. Indicate different regions of the phase plane that correspond to different types of asymptotic behavior.

Solution

- (a) The equilibria satisfy

$$x - y = 0, \qquad x + y - 2xy = 0,$$

which implies that $x = y$ and $x - x^2 = 0$, so $(x, y) = (0, 0)$ or $(x, y) = (1, 1)$.

- (b) The Jacobian matrix of the system is

$$\begin{pmatrix} 1 & -1 \\ 1 - 2y & 1 - 2x \end{pmatrix}.$$

- The linearization at $(0, 0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues are $\lambda = 1 \pm i$, so $(0, 0)$ is an unstable spiral point.

- The linearization at $(1, 1)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues are $\lambda = \pm\sqrt{2}$, so $(1, 1)$ is a saddle point.

- (c) The phase plane is shown below. The stable and unstable manifolds of the saddle point are shown in green, and the nullclines in orange ($y_t = 0$) and magenta ($x_t = 0$).

- (d) The phase plane is divided into two parts by a curve that consists of the stable manifold of the saddle point — which includes the heteroclinic orbit from the spiral $(0,0)$ to the saddle $(1,1)$ — and the trajectory from the spiral $(0,0)$ such that $x(t) \to -\infty$, $y(t) \to 1/2$ as $t \to +\infty$.

- Trajectories below this curve (the region actually winds around the spiral point near $(0,0)$) approach the right part of the unstable manifold of the saddle point, and have $x(t) \to +\infty$, $y(t) \to 1/2$ as $t \to +\infty$; while points above this curve approach the left part of the unstable manifold of the saddle point, and have $x(t) \to -\infty$, $y(t) \to +\infty$ as $t \to +\infty$.