Problem set 7

Math 207A, Fall 2014

1. The Lennard-Jones potential

$$V(r) = \frac{1}{r^{12}} - \frac{2}{r^6}$$

provides a simple model for the potential energy of molecules separated by a distance r, with weak long-range attraction (e.g., due to van der Waals forces) and strong short-range repulsion (e.g., due to the Pauli exclusion principle). Sketch the (r, r_t)-phase portrait of the ODE, with $r(t) > 0$, for radial motion in this potential:

$$r_{tt} + V'(r) = 0.$$

2. The Korteweg-deVries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$

describes nonlinear dispersive waves.

(a) Look for traveling wave solutions $u(x,t) = f(x - ct)$, and show that f satisfies the ODE (where we set a constant of integration to 0 without loss of generality)

$$f'' + \frac{1}{2}f^2 - cf = 0.$$

(b) Sketch the (f, f')-phase plane of the ODE in (a) for: (i) $c < 0$; (ii) $c = 0$; (iii) $c > 0$. What types of bounded traveling wave solutions does the KdV equation have?

(c) Verify that the KdV equation has soliton traveling wave solutions of the form

$$u(x,t) = a \operatorname{sech}^2[k(x - ct)]$$

for suitable constants a, k. What orbit does this solution correspond to in your phase planes from (b)?

3. Consider a planar ODE of the following form with a quadratic vector field:

$$x_t = a_1 x^2 + b_1 xy + c_1 y^2 + \alpha_1 x + \beta_1 y,$$
$$y_t = a_2 x^2 + b_2 xy + c_2 y^2 + \alpha_2 x + \beta_2 y.$$

Solve these equations numerically for the following values of the coefficients:

$$a_1 = 1; \quad b_1 = 1; \quad c_1 = 0; \quad \alpha_1 = 0; \quad \beta_1 = 1;$$
$$a_2 = -10; \quad b_2 = 2.2; \quad c_2 = 0.7; \quad \alpha_2 = -72.7778; \quad \beta_2 = 0.0015.$$

Plot some trajectories in "small" and "large" regions of the phase plane with:

(a) $-5 \le x \le 20, \quad -80 \le y \le 40;$

(b) $-4500 \le x \le 0, \quad -6000 \le y \le 14000.$

How many limit cycles can you find?

4. Consider the planar system

$$x_t = \mu x - y + x^2, \qquad y_t = x + \mu y + x^2.$$

Use the Hopf bifurcation theorem to show that a Hopf bifurcation occurs at $\mu = 0$. Plot numerically a phase portrait for $\mu = -0.1, 0.1, 0.2$. Is the Hopf bifurcation subcritical or supercritical? What happens to the limit cycle?