METHODS OF APPLIED MATHEMATICS
Math 207A, Fall 2018
Final: Solutions

1 [10pts] Suppose that the vector field in a planar dynamical system

$$x_t = f(x, y), \qquad y_t = g(x, y)$$

satisfies

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} < 0 \qquad \text{for all } (x, y) \in \mathbb{R}^2. \tag{1}$$

Show that the system cannot have any closed periodic orbits. HINT. Recall Green's theorem: If $\Omega \subset \mathbb{R}^2$ is a subset of the plane whose boundary $\partial\Omega$ is a smooth, simple closed curve and $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions, then

$$\int_{\partial\Omega} P dy - Q dx = \int_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy.$$

Solution

- Suppose, for contradiction, that Γ is a closed orbit. Then Γ is a smooth simple closed curve (assuming, as usual, that f, g are smooth functions) with interior Ω. We parametrize Γ by time $0 \le t \le T$, so that (with a slight abuse of notation) it is given by $x = x(t)$, $y = y(t)$ where $(x(t), y(t))$ are solutions of the differential equation. It follows that

$$\int_{\Gamma} (f dy - g dx) = \int_0^T (f, g) \cdot (y_t, -x_t)\, dt = \int_0^T (fg - gf)\, dt = 0.$$

- On the other hand, Green's theorem and (1) imply that

$$\int_{\Gamma} (f dy - g dx) = \int_{\Omega} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dxdy < 0,$$

  which shows that periodic orbits are impossible.

Remark. This result is called Bendixson's criterion. It would apply equally well if the divergence of $\mathbf{f} = (f, g)$ was strictly positive. More generally, Dulac's criterion states that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^2$ cannot have any closed orbits if there exists a strictly positive function h such that $\nabla \cdot (h\mathbf{f})$ has a definite sign. The divergence of $\mathbf{f}$ is related to how the flow changes areas (or volumes) in phase space (the flow decreases, increases, or preserves volumes if $\nabla \cdot \mathbf{f}$ is negative, positive, or zero, respectively).

2 [20pts] Consider the planar system

$$x_t = -x + y^2, \qquad y_t = -2x^2 + 2xy^2.$$

(a) Determine the equilibria, find an equation for the trajectories of the system, and sketch the phase plane.

(b) Linearize the system at $(x, y) = (0, 0)$ and determine the stable and center subspaces. What are the stable and center manifolds of $(0, 0)$?

Solution

- (a) Every point on the parabola $x = y^2$ is an equilibrium. For $x \neq y^2$, we have on trajectories that

  $$\frac{dy}{dx} = \frac{-2x^2 + 2xy^2}{-x + y^2} = 2x.$$

  Integration of this ODE shows that the trajectories satisfy

  $$y = x^2 + C$$

  for some constant C. The solution for $x(t)$ is increasing when $x < y^2$ and decreasing when $x > y^2$. The phase plane is shown below. In particular, there is a heteroclinic orbit $y = x^2$ with $0 < x < 1$ that connects $(1, 1)$ to $(0, 0)$.

- (b) The linearization at $(0, 0)$ is

  $$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  The stable subspace is spanned by the eigenvector $(1, 0)^T$ with eigenvalue -1, and the center subspace is spanned by the eigenvector $(0, 1)^T$ with eigenvalue 0.

- The local stable and center manifolds are the curves $y = x^2$ and $x = y^2$ in a small neighborhood of the origin. The global stable manifold, obtained by mapping the local stable manifold backward in time, is

  $$W^s(0, 0) = \left\{(x, y) \in \mathbb{R}^2 : y = x^2 \text{ and } -\infty < x < 1\right\}.$$

  We could also use the whole of the other parabola as a center manifold,

  $$W^c(0, 0) = \left\{(x, y) \in \mathbb{R}^2 : x = y^2 \text{ and } -\infty < y < -\infty\right\}.$$

3 [20pts] Consider the following flow on the circle

$$\theta_t = 1 - \mu \sin\theta,$$

where $\theta(t) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is an angle and $\mu > 0$ is a parameter.

(a) Find the equilibria and determine their stability as a function of μ. Sketch a bifurcation diagram. What is the bifurcation point (θ_*, μ_*)? What type of bifurcation occurs?

(b) Sketch the flows on the phase circle for $0 < \mu < \mu_*$, $\mu = \mu_*$, and $\mu > \mu_*$.

Solution

- (a) The equilibria satisfy $\sin\theta = 1/\mu$. For $0 < \mu < 1$, there are no equilibria; for $\mu = 1$, there is a single equlibrium at $\theta = \pi/2$; and for $\mu > 1$, there are two equilibria at $\theta = \pi/2 \pm \phi(\mu)$, where $0 < \phi < \pi/2$ satisfies $\cos\phi = 1/\mu$.

- If $f(\theta, \mu) = 1 - \mu\sin\theta$, then $f_\theta(\theta, \mu) = -\mu\cos\theta$, so

  $$f_\theta\left(\frac{\pi}{2} \pm \phi\right) = \pm\mu\sin\phi.$$

  It follows that $f_\theta < 0$ at $\theta = \pi/2 - \phi$, so the equilibrium is asymptotically stable, and and $f_\theta > 0$ at $\theta = \pi/2 + \phi$, so the equilibrium is unstable. (Alternatively, we can look at the sign of f to determine the direction of the flow.)

- A saddle-node bifurcation occurs at $(\theta, \mu) = (\pi/2, 1)$. The bifurcation diagram is shown below.

- (b) Phase flows are shown below.

4 [20pts] The Ricker model for a population x_n at generation $n = 0, 1, 2, \ldots$ is

$$x_{n+1} = x_n \exp\left[\mu\left(1 - x_n\right)\right],$$

where $-\infty < \mu < \infty$ is a growth rate parameter.

(a) Find the fixed points and determine their stability.

(b) Sketch a bifurcation diagram for the fixed points and discuss what bifurcations occur at the fixed points as μ increases from $-\infty$ to ∞.

Solution

- (a) The fixed points satisfy $x = x \exp\left[\mu\left(1 - x\right)\right]$, so either $x = 0$ or

  $$\exp\left[\mu\left(1 - x\right)\right] = 1.$$

  This equation implies that $\mu(1 - x) = 0$, so either $x = 1$ or $\mu = 0$. Thus, for $\mu \neq 0$ there are two fixed points $x = 0, 1$, and for $\mu = 0$ every point is a fixed point.

- If $f(x, \mu) = x \exp\left[\mu\left(1 - x\right)\right]$, then

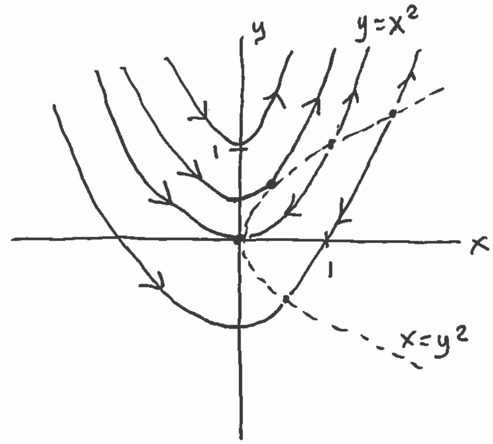
  $$f_x(x, \mu) = \exp\left[\mu\left(1 - x\right)\right] - \mu x \exp\left[\mu\left(1 - x\right)\right].$$

  It follows that $f_x(0, \mu) = e^{\mu}$, so $x = 0$ is asymptotically stable for $\mu < 0$, when $0 < f_x(0, \mu) < 1$, and unstable for $\mu > 0$, when $f_x(0, \mu) > 1$. Similarly, $f_x(1, \mu) = 1 - \mu$, so $x = 1$ is unstable for $\mu < 0$ or $\mu > 2$, and asymptotically stable for $0 < \mu < 2$.

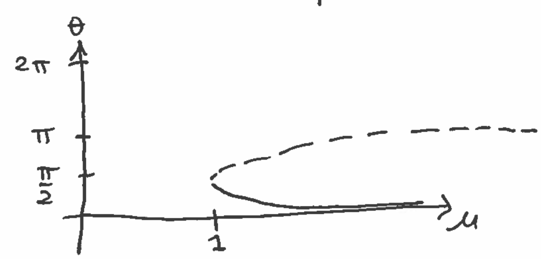
- At $\mu = 0$, the fixed points are nonhyperbolic, with $f_x(x, 0) = 1$, so linearized stability does not tell us their stability. However, since every point is a fixed point, all of the fixed points are stable but not asymptotically stable.

- (b) At $(x, \mu) = (0, 0)$ and $(x, \mu) = (1, 0)$ there is a kind of degenerate or critical transcritical bifurcation, in which the fixed-point branches $x = 0$, $x = 1$ cross the branch at $\mu = 0$. When this happens, the branch $x = 0$ loses stability and the branch $x = 1$ gains stability. At $(x, \mu) = (1, 2)$, the branch $x = 1$ loses stability as the corresponding eigenvalue decreases through -1, so we expect a period-doubling bifurcation to occur. A more detailed analysis of the nonlinear terms, using a Taylor expansion of $f(x, \mu)$ around $(x, \mu) = (1, 2)$, shows that there is a super-critical period-doubling bifurcation. The bifurcation diagram is sketched below.

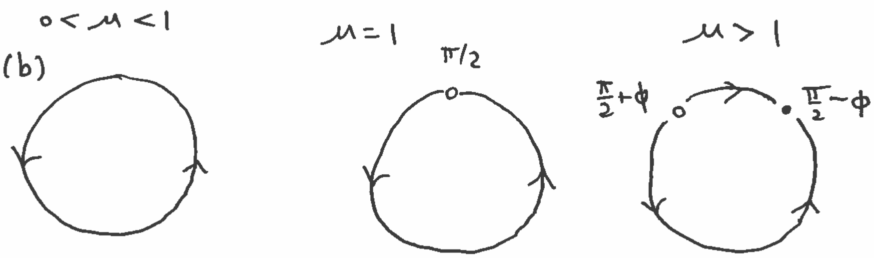
2(a)


y
$y=x^2$
1
x
1
$x=y^2$

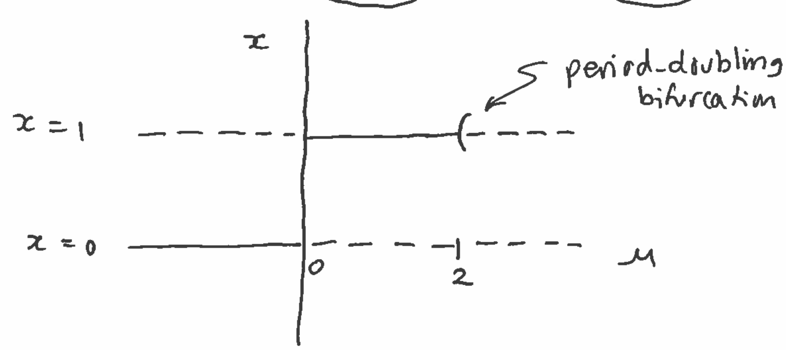
3(a)


θ
2π
π
π/2
1
μ

3(b)


$0<\mu<1$
$\mu=1$
π/2
$\mu>1$
$\frac{\pi}{2}+\phi$
$\frac{\pi}{2}-\phi$

4(b)


x
period-doubling bifurcation
$x=1$
$x=0$
0
2
μ