Problem set 1
Math 207A, Fall 2018
Due: Fri., Oct. 5

1. Consider the ODE for $x(t) \in \mathbb{R}$ given by

$$\dot{x} = x \log |x|.$$

(a) Compute the flow map $\varphi_t : \mathbb{R} \to \mathbb{R}$. (We define $x \log |x| = 0$ for $x = 0$.)

(b) Use your solution in (a) to verify explicitly that φ_t satisfies the group property $\varphi_s \circ \varphi_t = \varphi_{s+t}$, and find the fixed points of φ_t.

2. Define $E : \mathbb{R}^2 \to \mathbb{R}$ by

$$E(x, y) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

(a) Sketch the phase plane of the Hamiltonian system

$$\dot{x} = \frac{\partial E}{\partial y}, \qquad \dot{y} = -\frac{\partial E}{\partial x},$$

and discuss the stability of the equilibria.

(a) Sketch the phase plane of the gradient system

$$\dot{x} = -\frac{\partial E}{\partial x}, \qquad \dot{y} = -\frac{\partial E}{\partial y},$$

and discuss the stability of the equilibria.

3. The following Hamiltonian, depending on $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$, describes two decoupled simple harmonic oscillators, one with positive energy, the other with negative energy:

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\left(q_1^2 + p_1^2\right) - \frac{1}{2}\left(q_2^2 + p_2^2\right).$$

(a) Write down Hamilton's equations and solve them. Deduce that the equilibrium $(q_1, q_2, p_1, p_2) = (0, 0, 0, 0)$ is stable. What kind of critical point does H have at this equilibrium?

(b) Suppose we include an interaction term in the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\left(q_1^2 + p_1^2\right) - \frac{1}{2}\left(q_2^2 + p_2^2\right) + kq_1q_2,$$

where $k \in \mathbb{R}$ is a constant. What happens to the stability of the equilibrium?

4. Consider the Lorenz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z, \end{aligned}$$

with parameter values $\sigma = 10$, $\beta = 8/3$, $r = 28$.

(a) Solve the Lorenz equations numerically with initial conditions

$$x(0) = -2, \qquad y(0) = -4, \qquad z(0) = 12$$

for $0 \leq t \leq 30$. Plot the trajectory of this solution in (x, y, z)-phase space, and plot the graph of $x(t)$ versus t.

(b) Solve the Lorenz equations numerically with initial conditions

$$x(0) = -2.0001, \qquad y(0) = -4, \qquad z(0) = 12$$

for $0 \leq t \leq 30$, and plot the graph of $x(t)$ versus t on the same plot as the one from (a).

HINT. In MATLAB, use `ode45` to solve the ODE, `plot3` to plot the trajectory in phase space, and `plot` to plot the graphs of $x(t)$.