PROBLEM SET 1: Solutions
Math 207A, Fall 2018

1. Consider the ODE for $x(t) \in \mathbb{R}$ given by

$$\dot{x} = x \log |x|.$$

(a) Compute the flow map $\varphi_t : \mathbb{R} \to \mathbb{R}$. (We define $x \log |x| = 0$ for $x = 0$.)
(b) Use your solution in (a) to verify explicitly that φ_t satisfies the group property $\varphi_s \circ \varphi_t = \varphi_{s+t}$, and find the fixed points of φ_t.

Solution

- (a) If $x(t) \neq 0, \pm 1$, then separation of variables in the ODE and the substitution $u = \log |x|$ gives

$$\int \frac{dx}{x \log |x|} = \int dt \implies \int \frac{du}{u} = t + C \implies \log |u| = t + C.$$

If $x(0) = x_0$, it follows that

$$x(t) = (\operatorname{sgn} x_0) e^{\log |x_0| e^t} = (\operatorname{sgn} x_0) |x_0|^{e^t}.$$

We also have the equilibrium solutions $x(t) = 0, \pm 1$.

- The flow map $\varphi_t : \mathbb{R} \to \mathbb{R}$ is therefore given by

$$\varphi_t(x) = (\operatorname{sgn} x)|x|^{e^t} \quad \text{if } x \neq 0, \qquad \varphi_t(0) = 0.$$

- (b) If $x > 0$, then

$$\varphi_t \circ \varphi_s(x) = \varphi_t\left(x^{e^s}\right) = \left(x^{e^s}\right)^{e^t} = x^{e^{s+t}} = \varphi_{s+t}(x),$$

and similarly for $x < 0$ and $x = 0$.

- The fixed points of φ_t, such that $\varphi_t(x) = x$, are given by $x = 0, \pm 1$.

2. Define $E : \mathbb{R}^2 \to \mathbb{R}$ by

$$E(x, y) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

(a) Sketch the phase plane of the Hamiltonian system $\dot{x} = \frac{\partial E}{\partial y}$, $\dot{y} = -\frac{\partial E}{\partial x}$. and discuss the stability of the equilibria.

(b) Sketch the phase plane of the gradient system $\dot{x} = -\frac{\partial E}{\partial x}$, $\dot{y} = -\frac{\partial E}{\partial y}$, and discuss the stability of the equilibria.

Solution

- (a) Trajectories of the Hamiltonian system $\dot{x} = y$, $\dot{y} = x - \frac{1}{2}x^2$ are the level curves of E, in the direction of increasing x for $y > 0$. The equilibrium $(x, y) = (0, 0)$ is unstable (a saddle point). The equilibrium $(x, y) = (2, 0)$, where V has a local minimum, is stable but not asymptotically stable (a center).

- (b) Trajectories of the gradient system $\dot{x} = x - \frac{1}{2}x^2$, $\dot{y} = -y$ are orthogonal to the level curves of E, in the direction of decreasing E. The equilibrium $(x, y) = (0, 0)$ is unstable (a saddle point). The equilibrium $(x, y) = (2, 0)$ is asymptotically stable (a stable node).

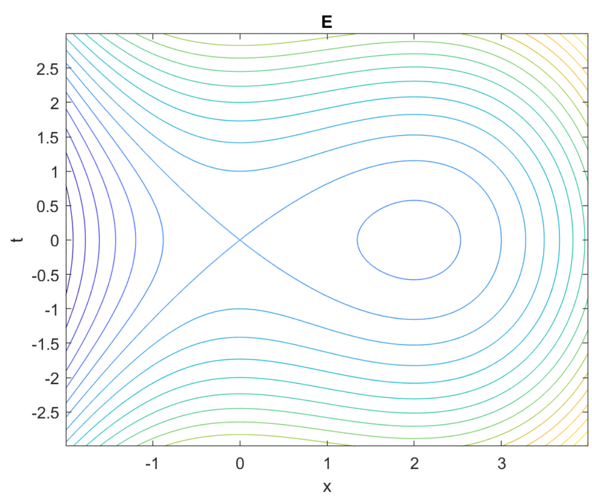

Figure 1: The level curves of E. Phase planes are shown on the next page.

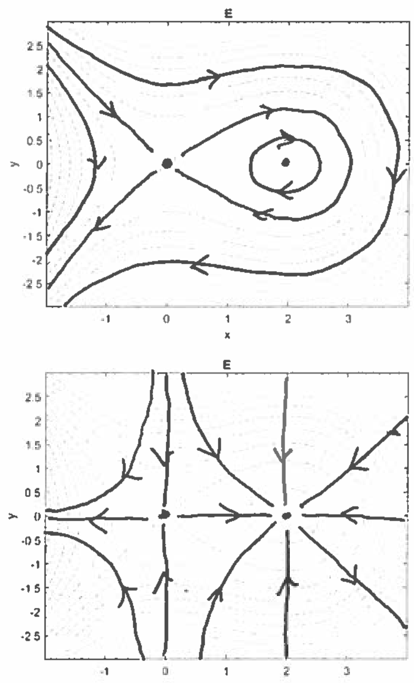

Figure 2: Top: (a) Hamiltonian system. Bottom: (b) Gradient system.

3. The following Hamiltonian, depending on $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$, describes two decoupled simple harmonic oscillators, one with positive energy, the other with negative energy:

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\left(q_1^2 + p_1^2\right) - \frac{1}{2}\left(q_2^2 + p_2^2\right).$$

(a) Write down Hamilton's equations and solve them. Deduce that the equilibrium $(q_1, q_2, p_1, p_2) = (0, 0, 0, 0)$ is stable. What kind of critical point does H have at this equilibrium?

(b) Suppose we include an interaction term in the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\left(q_1^2 + p_1^2\right) - \frac{1}{2}\left(q_2^2 + p_2^2\right) + kq_1q_2,$$

where $k \in \mathbb{R}$ is a constant. What happens to the stability of the equilibrium?

Solution

- (a) Hamilton's equations are

  $$\dot{q}_1 = p_1, \quad \dot{p}_1 = -q_1, \qquad \dot{q}_2 = -p_2, \quad \dot{p}_2 = q_2.$$

  These equations are linear since the Hamiltonian is quadratic. They are a pair of decoupled Hamiltonian equations for (q_1, p_1) and (q_2, p_2).

- It follows that

  $$\frac{1}{2}\left(q_1^2 + p_1^2\right) = E_1, \qquad \frac{1}{2}\left(q_2^2 + p_2^2\right) = E_2, \tag{1}$$

  where E_1, E_2 are constants. In particular

  $$q_1^2 + q_2^2 + p_1^2 + p_2^2 = 2(E_1 + E_2)$$

  is constant on solutions, which implies that the equilibrium $(0, 0, 0, 0)$ is stable.

- Elimination of the p_j gives two decoupled simple harmonic oscillator equations

  $$\ddot{q}_1 + q_1 = 0, \qquad \ddot{q}_2 + q_2 = 0,$$

with solutions

$$q_1(t) = a_1 \cos(t + \delta_1), \qquad p_1(t) = -a_1 \sin(t + \delta_1),$$
$$q_2(t) = a_2 \cos(t + \delta_2), \qquad p_2(t) = a_2 \sin(t + \delta_2),$$

where the a_j, δ_j are constants, which also shows that the equilibrium $(0,0,0,0)$ is stable.

- The equilibrium is a saddle point of H. Nevertheless, it is stable because of the additional conserved quantities in (1).

- (b) Hamilton's equations are

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -q_1 - kq_2, \qquad \dot{q}_2 = p_2, \quad \dot{p}_2 = -q_2 - kq_1.$$

Elimination of variables gives

$$q_1^{(4)} + 2\ddot{q}_1 + \left(1 + k^2\right) q_1 = 0.$$

Looking for solutions $q_1(t) = e^{rt}$, we get the characteristic equation

$$r^4 + 2r^2 + 1 + k^2 = 0 \implies \left(r^2 + 1\right)^2 = -k^2,$$

so

$$r^2 = -1 \pm ik = \sqrt{1 + k^2} e^{\pm i\delta}, \qquad \delta = \arg(-1 + ik).$$

We then get 4 roots for r of the form $\pm a \pm ib$ where

$$a = \left(1 + k^2\right)^{1/4} \cos\left(\frac{\delta}{2}\right).$$

If $k \neq 0$, then $\delta \not\equiv \pi$, so $a \neq 0$. Hence, two of these roots have positive real part, meaning that there exist perturbations of the equilibrium that grow exponentially in time, so the equilibrium is unstable.

- The energy of the (q_1, p_1)-oscillator increases with the amplitude of the oscillations, while the energy of the (q_2, p_2)-oscillator decreases with the amplitude of the oscillations. When the oscillators can exchange energy, the amplitude of both oscillators can increase while conserving the total energy of the system.

4. Consider the Lorenz equations

$$x_t = \sigma(y - x), \qquad y_t = rx - y - xz, \qquad z_t = xy - \beta z,$$

with parameter values $\sigma = 10$, $\beta = 8/3$, $r = 28$.

(a) Solve the Lorenz equations numerically with initial conditions $x(0) = -2$, $y(0) = -4$, $z(0) = 12$ for $0 \leq t \leq 30$. Plot the trajectory of this solution in (x, y, z)-phase space, and plot the graph of $x(t)$ versus t.

(b) Solve the Lorenz equations numerically with initial conditions $x(0) = -2.0001$, $y(0) = -4$, $z(0) = 12$ for $0 \leq t \leq 30$, and plot the graph of $x(t)$ versus t on the same plot as the one from (a).

Solution

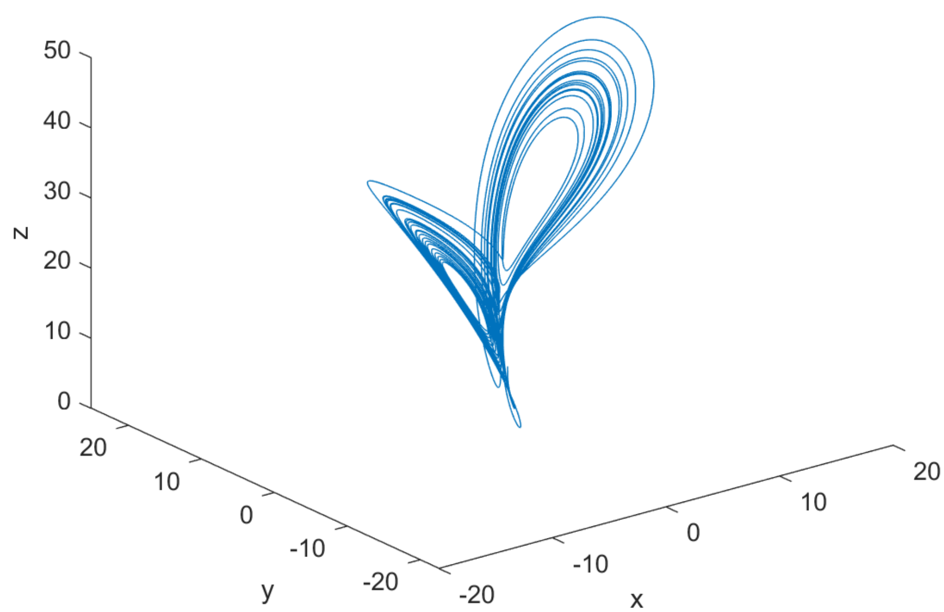

Figure 2: (a) Solution trajectory in phase space. The trajectory is winding around the Lorenz attractor, which has a fractal structure, and jumping unpredictably from one "wing" to the other.

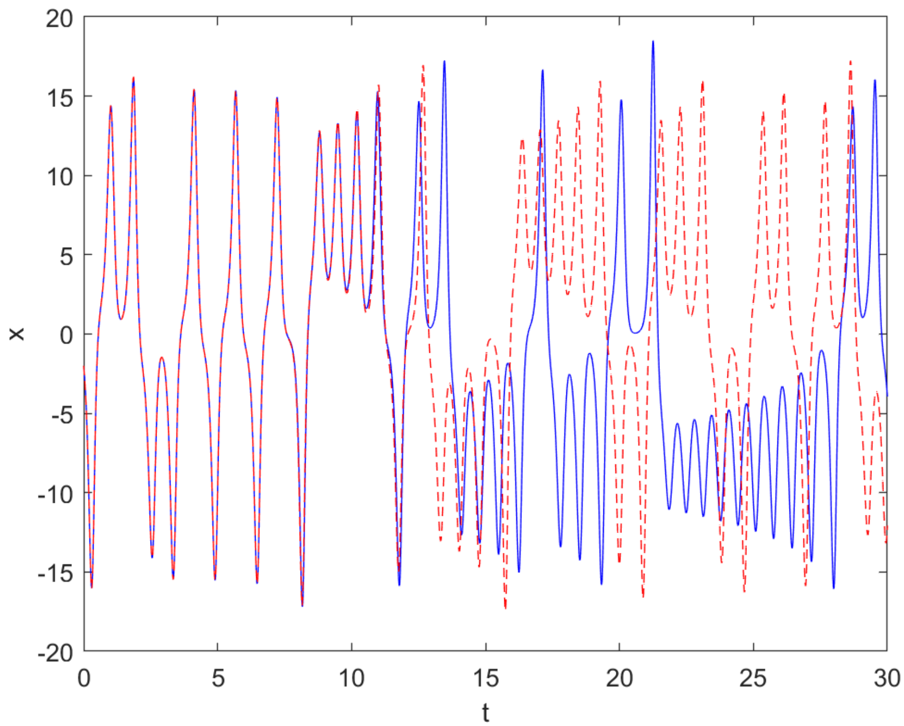

Figure 3: (b) Blue: solution for $x(t)$ with $x(0) = -2$. Red: solution for $x(t)$ with $x(0) = -2.0001$. Note the sensitive dependence on initial data.