PROBLEM SET 2
Math 207A, Fall 2018
Due: Fri., Oct. 12

1. Suppose that $x(t) = \cos t$ is a solution of the autonomous, scalar ODE $x_t = f(x)$ for some smooth function $f : \mathbb{R} \to \mathbb{R}$. Show that $x(t) = -\sin t$ is also a solution.

2. Prove that a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. If, in addition, there exists a constant $M \geq 0$ such that $|\partial f/\partial x| \leq M$ for all $x \in \mathbb{R}^n$, prove that f is globally Lipscitz continuous.

HINT. Note that

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f\left(tx + (1-t)y\right)\, dt.$$

3. Compute the Picard iterates for the following scalar initial value problems, and discuss their convergence:

$$(a)\ x_t = x, \quad x(0) = 1; \qquad (b)\ x_t = 2t - 2\sqrt{\max(0,x)}, \quad x(0) = 0.$$

4. Consider the following initial value problem for $x : \mathbb{R} \to \mathbb{R}$

$$x_t + x = \cos t, \qquad x(0) = x_0.$$

(a) How would you classify this ODE? What do general theorems say about the local/global existence and uniqueness of solutions?

(b) Define a Poincaré map $P : \mathbb{R} \to \mathbb{R}$ by $P(x_0) = x(2\pi)$, where $x(t)$ is the solution in (a). Compute P and find its fixed point. Show that the fixed point of P corresponds to a 2π-periodic solution of the original ODE. Discuss the stability of this solution.

5. Consider the following 2×2-system for $(x(t), y(t))$:

$$x_t = x - y - x^3, \qquad y_t = x + y - y^3.$$

(a) What do general theorems say about the local/global existence and uniqueness of solutions of the initial value problem with $x(0) = x_0$, $y(0) = y_0$?

(b) Let $V(x,y) = x^2 + y^2$. Compute

$$\frac{d}{dt} V\left(x(t), y(t)\right)$$

and use the result to show that the solution of the initial value problem exists globally forwards in time for all $t \geq 0$.

(c) Let $0 < a < 1$ and $b > 2$. If $(x_0, y_0) \neq (0,0)$, show that the solution satisfies

$$a \leq x^2(t) + y^2(t) \leq b$$

for all sufficiently large $t > 0$. Do you have any guesses for the long time behavior of the solution?