PROBLEM SET 2: Solutions
Math 207A, Fall 2018

1. Suppose that $x(t) = \cos t$ is a solution of the autonomous, scalar ODE $x_t = f(x)$ for some smooth function $f : \mathbb{R} \to \mathbb{R}$. Show that $x(t) = -\sin t$ is also a solution.

Solution

- If $x(t)$ is a solution of an autonomous equation, then $y(t) = x(t+c)$ is also a solution for any constant c, since

  $$y_t(t) = x_t(x+c) = f\left(x(t+c)\right) = f\left(y(t)\right).$$

  This argument doesn't work for a nonautonomous equation $x_t = f(x,t)$, since then

  $$y_t(t) = x_t(t+c) = f\left(x(t+c), t+c\right) = f\left(y(t), t+c\right) \neq f\left(y(t), t\right).$$

- We have $-\sin t = \cos(t + \pi/2)$, so the result follows from (a).

2. Prove that a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. If, in addition, there exists a constant $M \geq 0$ such that $|\partial f_i/\partial x_j| \leq M$ for all $x \in \mathbb{R}^n$ and $1 \leq i, j \leq n$, prove that f is globally Lipschitz continuous.

HINT. Note that

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f\left(tx + (1-t)y\right)\, dt.$$

Solution

- Given a norm $|x|$ of vectors $x \in \mathbb{R}^n$, we define the corresponding norm of $n \times n$ matrices A by (see e.g. §3.1 of Teschl)

$$\|A\| = \max_{x \neq 0} \frac{|Ax|}{|x|}.$$

  In particular, $|Ax| \leq \|A\|\, |x|$ for any vector $x \in \mathbb{R}^n$.

- The hint follows directly from the fundamental theorem of calculus:

$$\int_0^1 \frac{d}{dt} f\left(tx + (1-t)y\right)\, dt = \left[f\left(tx + (1-t)y\right)\right]_{t=0}^{t=1} = f(x) - f(y).$$

- We have

$$\frac{d}{dt} f\left(tx + (1-t)y\right) = Df\left(tx + (1-t)y\right)(x - y),$$

  where $Df = (\partial f_i/\partial x_j)$ is the Jacobian matrix of $f = (f_1, \ldots, f_n)$. The component form of this equation is

$$\frac{d}{dt} f_i\left(tx + (1-t)y\right) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\left(tx + (1-t)y\right)(x_j - y_j).$$

  It follows that

$$\left|\frac{d}{dt} f\left(tx + (1-t)y\right)\right| \leq \left\|Df\left(tx + (1-t)y\right)\right\|\, |x - y|.$$

- If f is continuously differentiable, then the components of Df are continuous and therefore uniformly bounded on any convex, compact set $K \subset \mathbb{R}^n$, so $\|Df\| \leq M$ on K for some constant M. It follows that

$$\begin{aligned}
|f(x) - f(y)| &\leq \int_0^1 \left| \frac{d}{dt} f\left(tx + (1-t)y\right) \right| dt \\
&\leq \int_0^1 \|Df\left(tx + (1-t)y\right)\| \, |x - y| \, dt \\
&\leq M|x - y|
\end{aligned}$$

  for all $x, y \in K$, which shows that f is locally Lipschitz continuous.

- If the partial derivatives of f are uniformly bounded on $\mathbb{R}^n$, then the previous estimate holds for all $x, y \in \mathbb{R}^n$, so f is globally Lipschitz continuous.

3. Compute the Picard iterates for the following scalar initial value problems, and discuss their convergence:

$$(a)\ x_t = x, \quad x(0) = 1; \qquad (b)\ x_t = 2t - 2\sqrt{\max(0, x)}, \quad x(0) = 0.$$

Solution

- (a) The nth iterate with $x_t^{n+1} = x^n$ and $x^0 = 1$ is given by
$$x^n(t) = \sum_{k=0}^{n} \frac{t^k}{k!}.$$

- This result follows by induction. It holds for $n = 0$, and if the result holds for some $n \geq 0$, then
$$x_t^{n+1} = \sum_{k=0}^{n} \frac{t^k}{k!}, \qquad x^{n+1}(0) = 1,$$
which implies the result for $n + 1$.

- The Picard iterates $x^n(t)$ are the Taylor polynomials of the solution e^t. They converge pointwise (and uniformly on compact sets) to the solution on $\mathbb{R}$.

- (b) We consider the cases $t \geq 0$ and $t \leq 0$ separately. If $t \geq 0$, then
$$x^n(t) = \begin{cases} 0 & \text{for even } n, \\ t^2 & \text{for odd } n. \end{cases}$$
The iterates do not converge on $[0, \infty)$, but oscillate between 0 and the solution t^2 of the initial value problem forward in time. (This result does not contradict the Picard theorem because the right hand side of the ODE is not Lipschitz continuous in x.)

- If $t \leq 0$, then $x^n(t) = a_n t^2$ where $a_0 = 0$, $a_1 = 1$, and
$$a_{n+1} = 1 + \sqrt{a_n} \qquad n \geq 0. \tag{1}$$
Since $1 \leq a_n \leq 3$ implies that $1 \leq a_{n+1} \leq 3$, and $a_n - a_{n-1} > 0$ implies that
$$a_{n+1} - a_n = \sqrt{a_n} - \sqrt{a_{n-1}} > 0,$$
it follows by induction that (a_n) is an increasing sequence of positive numbers that is bounded from above by 3.

- Bounded monotone sequences converge, so $a_n \to a$ as $n \to \infty$ for some $1 \leq a \leq 3$. Taking the limit of (1), we find that $a = 1 + \sqrt{a}$, which implies that
$$a = \frac{3 + \sqrt{5}}{2}.$$
- It follows that the Picard iterates converge pointwise on $(-\infty, 0]$, and uniformly on compact sets, to the solution $x(t) = at^2$ of the final value problem backward in time.

4. Consider the following initial value problem for $x : \mathbb{R} \to \mathbb{R}$

$$x_t + x = \cos t, \qquad x(0) = x_0.$$

(a) How would you classify this ODE? What do general theorems say about the local/global existence and uniqueness of solutions?
(b) Define a Poincaré map $P : \mathbb{R} \to \mathbb{R}$ by $P(x_0) = x(2\pi)$, where $x(t)$ is the solution in (a). Compute P and find its fixed point. Show that the fixed point of P corresponds to a 2π-periodic solution of the original ODE. Discuss the stability of this solution.

Solution

- (a) The ODE is first order, scalar, linear, constant coefficient, and nonhomogeneous. The general theorem for linear equations with continuous coefficients and nonhomogeneous term implies that there is a unique global solution.

- Alternatively, in order to use the Picard theorem stated in class, we can write the equation as a 2×2 autonomous system for (x, s) where $s = t$ and
$$x_s = -x + \cos s, \qquad s_t = 1.$$
The vector field $f(x, s) = (-x + \cos s, 1)$ is continuously differentiable with uniformly bounded derivatives on $\mathbb{R}^2$, so it is globally Lipschitz, and the Picard theorem implies that there is a unique global solution.

- (b) Using an imtegrating factor e^t, we get that
$$\frac{d}{dt}\left(e^t x\right) = e^t \cos t, \qquad x(0) = x_0.$$
Integration of this equation and imposition of the initial condition gives
$$x(t) = \left(x_0 - \frac{1}{2}\right) e^{-t} + \frac{1}{2}\left(\cos t + \sin t\right).$$

- It follows that $P : x_0 \mapsto x(2\pi)$ is given by
$$P(x_0) = \left(x_0 - \frac{1}{2}\right) e^{-2\pi} + \frac{1}{2}.$$
The fixed point of $P(x_0)$ is $x_0 = 1/2$, which corresponds to the periodic solution $x(t) = (\cos t + \sin t)/2$.

- The periodic solution is globally asymptotically stable, in the sense that

$$x(t) \to \frac{1}{2}(\cos t + \sin t) \qquad \text{as } t \to \infty$$

for any $x_0 \in \mathbb{R}$. Equivalently, the fixed point $1/2$ is a globally asymptotically stable fixed point of the discrete dynamical system $x_{n+1} = P(x_n)$, in the sense that

$$P^n(x_0) \to \frac{1}{2} \qquad \text{as } n \to \infty$$

for any $x_0 \in \mathbb{R}$.

5. Consider the following 2×2-system for $(x(t), y(t))$:

$$x_t = x - y - x^3, \qquad y_t = x + y - y^3. \tag{2}$$

(a) What do general theorems say about the local/global existence and uniqueness of solutions of the initial value problem with $x(0) = x_0$, $y(0) = y_0$?
(b) Let $V(x, y) = x^2 + y^2$. Compute

$$\frac{d}{dt} V\left(x(t), y(t)\right)$$

and use the result to show that the solution of the initial value problem exists globally forwards in time for all $t \geq 0$.
(c) Let $0 < a < 1$ and $b > 2$. If $(x_0, y_0) \neq (0, 0)$, show that the solution satisfies

$$a < x^2(t) + y^2(t) < b$$

for all sufficiently large $t > 0$. Do you have any guesses for the long time behavior of the solution?

Solution

- (a) The vector field $f(x, y) = (x - y - x^3, x + y - y^3)$ is continuously differentiable, so the initial value problem has unique local solutions.

- We compute that

$$\begin{aligned}\frac{d}{dt} V\left(x(t), y(t)\right) &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x\left(x - y - x^3\right) + 2y\left(x + y - y^3\right) \\ &= 2\left(x^2 + y^2 - x^4 - y^4\right).\end{aligned}$$

  Since

$$\begin{aligned}x^2 + y^2 - x^4 - y^4 &= x^2 + y^2 - \frac{1}{2}\left(x^2 + y^2\right)^2 - \frac{1}{2}\left(x^2 - y^2\right)^2 \\ &\leq x^2 + y^2 - \frac{1}{2}\left(x^2 + y^2\right)^2\end{aligned}$$

  we get that

$$\frac{dV}{dt} \leq -V(V - 2). \tag{3}$$

  where we use the abbreviated notation $V = V\left(x(t), y(t)\right)$.

- It follows that V is a nonincreasing function of time whenever $V \geq 2$. Choosing $c \geq 2$ such that $V(x_0, y_0) \leq c$, we get that $V(x(t), y(t)) \leq c$ for all $t \geq 0$. The solution therefore remains bounded, and the extension theorem implies that it exists for all $t \geq 0$.

- (c) Since

$$\begin{aligned} x^2 + y^2 - x^4 - y^4 &= x^2 + y^2 - \left(x^2 + y^2\right)^2 + 2x^2y^2 \\ &\geq x^2 + y^2 - \left(x^2 + y^2\right)^2 \end{aligned}$$

  we also see that

$$\frac{dV}{dt} \geq V(1 - V). \tag{4}$$

- If $0 < a < 1$, then (4) implies that $V_t > 0$ when $V = a$, and if $b > 2$, then (3) implies that $V_t < 0$ when $V = b$. It follows that all trajectories of the system with $V = a$ or $V = b$ enter the annulus $a < V(x, y) < b$, after which they cannot leave it forward in time. (The annulus is called a trapping region, or positively invariant set, for the flow.)

- If $V(x_0, y_0) > b$, we claim that $V(x(t), y(t)) = b$ for some sufficiently large $t > 0$, after which the trajectory enters the annulus $a < V < b$ and is trapped there. Suppose, for contradiction, that $V(x(t), y(t)) > b$ for all $t \geq 0$. Then (3) implies that $dV/dt < -\epsilon$ for all $t \geq 0$ where $\epsilon = b(b - 2) > 0$, and

$$V(x(t), y(t)) = V(x_0, y_0) + \int_0^t \frac{d}{ds} V(x(s), y(s))\, ds < V(x_0, y_0) - \epsilon t.$$

  It follows that $V(x(t), y(t)) \to -\infty$ as $t \to \infty$, and this contradiction proves the claim.

- If $0 < V(x_0, y_0) < a$, then a similar argument for V an increasing function of t shows that the trajectory must enter $a < V(x, y) < b$, which completes that proof that every nonzero solution is trapped in the annulus for all sufficiently large $t > 0$.

- One can verify (numerically if necessary) that $(x, y) = (0, 0)$ is the only equilibrium of the system. Thus, solutions in the annulus cannot approach an equilibrium solution as $t \to \infty$. According to the

Poincaré-Bendixson theorem (to be discussed later in the class), the only possibility left is that the solutions approach a periodic orbit. The resulting limit cycle is shown in Figure 1.

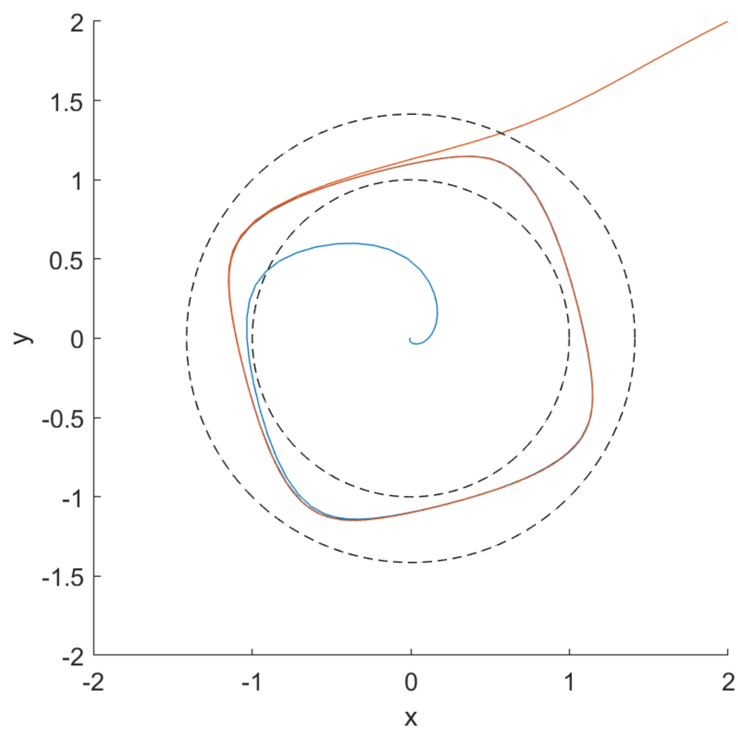

Figure 1: Phase plane for (2) showing the limit cycle solution. The dashed curves are the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 2$.