PROBLEM SET 3: Math 207A, Fall 2018
Due: Fri., Oct. 19

1. A model for a population $x(t) \geq 0$ with logistic growth and a constant rate of harvesting is

$$x_t = \mu x \left(1 - \frac{x}{K}\right) - H$$

where the parameters μ, K, H are positive constants.

(a) Show that a nondimensionalized form of the equation is

$$x_t = x(1 - x) - h, \tag{1}$$

and express the dimensionless parameter h as a ratio of two times.

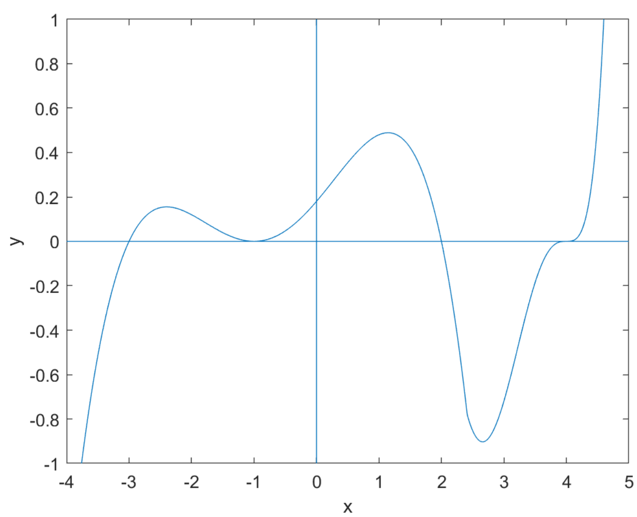
(b) Sketch a graph of the equilibria as functions of h, and sketch the phase line of (1) for various values of $h > 0$. Determine the stability of the equilibria, both from the phase line and from their linearized stability. For what values of the initial (nondimensionalized) population $x_0 > 0$ and harvesting rate $h > 0$ does the population become extinct?

2. The graph $y = f(x)$ of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is shown below. The function f has zeros only at certain integer values of x and is never zero outside the x-interval shown.

(a) Sketch the phase line for the ODE $x_t = f(x)$ and state the stability of the equilibria. Which of the equilibria are hyperbolic?

(b) Sketch the graph of the solution of the initial value problem with $x(0) = 0$.

(c) Sketch the graph of a potential $E(x)$ such that $f(x) = -E'(x)$.



3. A spherical raindrop with volume $V(t)$ and surface area $A(t)$ evaporates at a rate proportional to its surface area, meaning that $V_t = -kA$ for some constant $k > 0$. Write down an ODE for V and show that the raindrop evaporates completely in finite time. Find an expression for the evaporation time T in terms of k and the initial volume V_0 of the drop, and verify that your result is dimensionally consistent. Why doesn't this result violate the uniqueness part of the Picard theorem?

4. (a) Consider a scalar ODE $x_t = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the ODE cannot have a non-constant periodic solution with minimal period $T > 0$ such that $x(t + T) = x(t)$ for all $t \in \mathbb{R}$. Hint. Consider the integral

$$\int_0^T f(x)x_t\, dt,$$

(b) Why doesn't your argument in (a) apply to an ODE $\theta_t = f(\theta)$ on the circle $\mathbb{T}$?

5. (a) A Bernoulli equation is an ODE of the form

$$x_t = a(t)x + b(t)x^n$$

where a, b are continuous functions and $n \neq 1$. Show that the transformation

$$u = \frac{1}{x^{n-1}}$$

reduces a Bernoulli equation to a linear equation for u. Use this transformation to solve the logistic equation $x_t = x(1 - x)$.

(b) A Riccati equation is an ODE of the form

$$x_t = a(t) + b(t)x + c(t)x^2.$$

where a, b, c are continuous functions, with $c \neq 0$. Show that the transformation

$$x = -\frac{u_t}{cu}$$

reduces the Riccati equation to a second order, linear equation for u. Use this transformation to solve the logistic equation $x_t = x(1 - x)$.