Problem set 3: Solutions
Math 207A, Fall 2018

1. A model for a population $x(t) \geq 0$ with logistic growth and a constant rate of harvesting is

$$x_t = \mu x \left(1 - \frac{x}{K}\right) - H$$

where the parameters μ, K, H are positive constants.

(a) Show that a nondimensionalized form of the equation is

$$x_t = x(1-x) - h, \tag{1}$$

and express the dimensionless parameter h as a ratio of two times.

(b) Sketch a graph of the equilibria as functions of h, and sketch the phase line of (1) for various values of $h > 0$. Determine the stability of the equilibria, both from the phase line and from their linearized stability. For what values of the initial (nondimensionalized) population $x_0 > 0$ and harvesting rate $h > 0$ does the population become extinct?

Solution

- (a) Let P denote a unit of population and T denote a unit of time, then the parameters have dimensions

$$[\mu] = \frac{1}{T}, \qquad [K] = P, \qquad [H] = \frac{P}{T}.$$

- Define dimensionless variables based on logistic growth parameters,

$$\tilde{x} = \frac{x}{K}, \qquad \tilde{t} = \mu t.$$

Then

$$\tilde{x}_{\tilde{t}} = \tilde{x}(1-\tilde{x}) - \frac{H}{\mu K}.$$

Dropping tildes and letting $h = (1/\mu)/(K/H)$ yield (1).

- (b) To find equilibria, we solve the equation

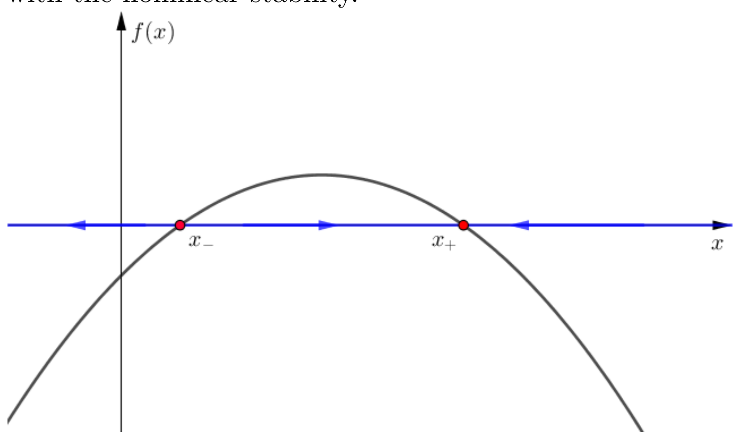
$$f(x) := x(1-x) - h = 0,$$

which gives

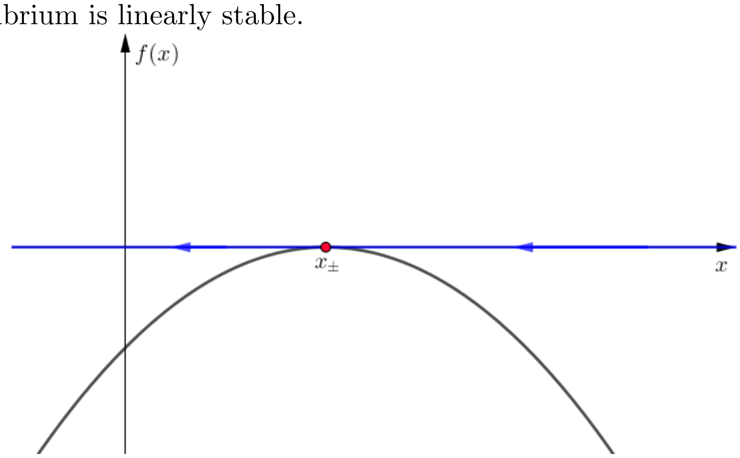
$$x_{\pm} = \frac{1 \pm \sqrt{1-4h}}{2}.$$

- The linearized equations of the nondimensionalized ODE around the equilibria (if exist) are
$$x_t = (1 - 2x_{\pm})x.$$

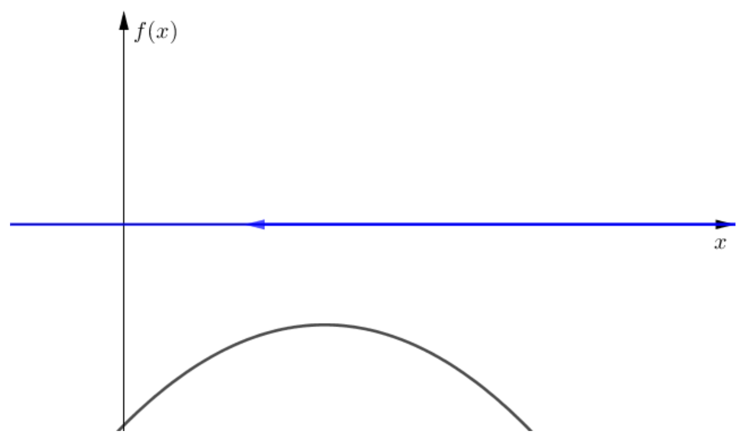
- When $0 < h < 1/4$, the equation admits two real solutions, the phase line (blue) and stability of equilibria can be determined based on the sign of $f(x)$. x_- is an unstable equilibrium, and x_+ is an asymptotically stable equilibrium. The two equilibria are hyperbolic, and the linearized stability agree with the nonlinear stability.



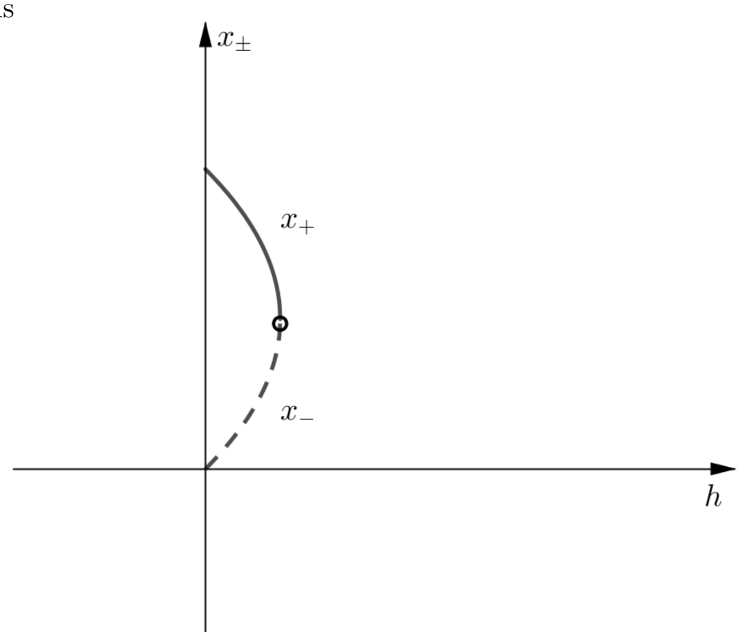
- When $h = 1/4$, the equation $f(x) = 0$ has two repeated solutions $x_{\pm} = 1/2$. The phase line (blue) is plotted as follows. The only equilibrium $x_{\pm}$ is semi-stable (unstable). The linearized equation in this case is $x_t = 0$, and the equilibrium is linearly stable.



- When $h > 1/4$, the equation $f(x) = 0$ admits no real solution. Therefore, there is no equilibrium point. The phase line is plotted as follows.



- In summary, if we use solid line to keep track of the stable nodes, and dashed line to keep track of the unstable nodes, a graph of equilibria as functions of h is



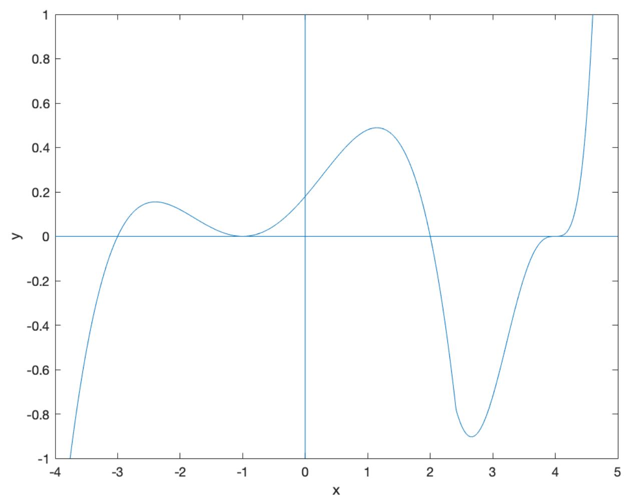
- It follows from the stability of equilibria that the population becomes extinct if $h > 1/4$, or $h < 1/4$ and $x_0 < x_-$.

2. The graph $y = f(x)$ of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is shown below. The function f has zeros only at certain integer values of x and is never zero outside the x-interval shown.

(a) Sketch the phase line for the ODE $x_t = f(x)$ and state the stability of the equilibria. Which of the equilibria are hyperbolic?

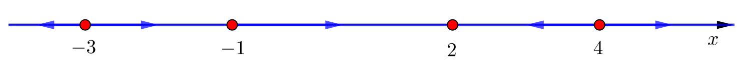
(b) Sketch the graph of the solution of the initial value problem with $x(0) = 0$.

(c) Sketch the graph of a potential $E(x)$ such that $f(x) = -E'(x)$.



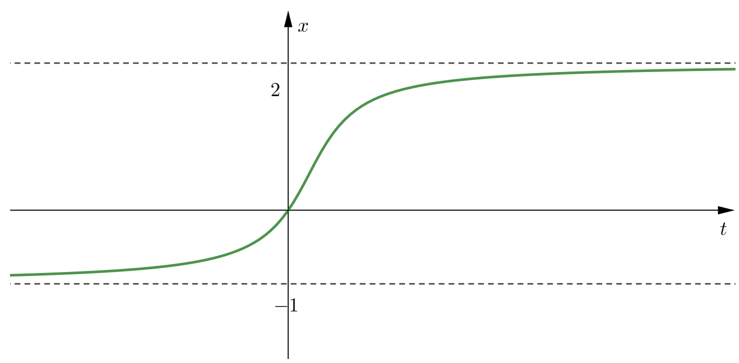
Solution

- (a) Based on the graph of f, the phase line for the ODE is

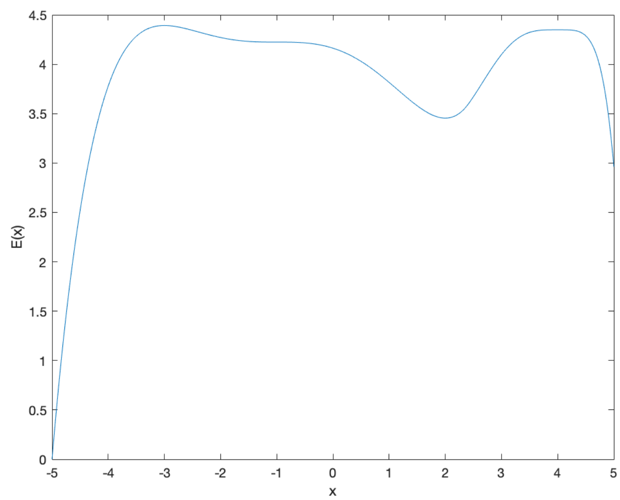


- We read from above graph that the equilibria 2 is asymptotically stable, -3 and 4 are unstable, and -1 is semi-stable (also unstable). Only -3 and 2 are hyperbolic as the derivatives of f there are nonzero.

- (b) A qualitative sketch of the solution with $x(0) = 0$ is



- (c) A sketch of the graph of potential $E(x)$ is as follows. A vertical translation of the graph is allowed, based on the choice of the base of the potential.



3. A spherical raindrop with volume $V(t)$ and surface area $A(t)$ evaporates at a rate proportional to its surface area, meaning that $V_t = -kA$ for some constant $k > 0$. Write down an ODE for V and show that the raindrop evaporates completely in finite time. Find an expression for the evaporation time T in terms of k and the initial volume V_0 of the drop, and verify that your result is dimensionally consistent. Why doesn't this result violate the uniqueness part of the Picard theorem?

Solution

- By the formula of surface area of 2-sphere and volume of 3-unit ball, we have the following equation
$$A = (36\pi)^{\frac{1}{3}} V^{\frac{2}{3}}.$$
- The ODE for V is then
$$V_t = -KV^{\frac{2}{3}},$$
where $K = (36\pi)^{\frac{1}{3}} k > 0$ is a constant.
- It is obvious that $V = 0$ is an equilibrium.
- Solving the Cauchy problem for the model with initial value $V(0) = V_0 > 0$ by separating variables yields
$$V(t) = \begin{cases} \left(\sqrt[3]{V_0} - \frac{K}{3}t\right)^3 & \text{if } t \leq \frac{3\sqrt[3]{V_0}}{K}, \\ 0 & \text{if } t > \frac{3\sqrt[3]{V_0}}{K}. \end{cases}$$
- The evaporation time $T = \frac{3\sqrt[3]{V_0}}{K}$. Let L denote a unit of length, and $\tilde{T}$ denote a unit of time, then
$$[K] = [k] = \frac{[V]}{[T] \cdot [A]} = \frac{L}{\tilde{T}}, \qquad \left[\frac{V^{\frac{1}{3}}}{K}\right] = \tilde{T} = [T],$$
which shows consistency.
- This result does not violate the uniqueness part of the the Picard theorem, as the $V^{\frac{2}{3}}$ is not Lipschitz continuous at $V = 0$.

4. (a) Consider a scalar ODE $x_t = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the ODE cannot have a non-constant periodic solution with minimal period $T > 0$ such that $x(t+T) = x(t)$ for all $t \in \mathbb{R}$. Hint. Consider the integral

$$\int_0^T f(x) x_t \, dt,$$

(b) Why doesn't your argument in (a) apply to an ODE $\theta_t = f(\theta)$ on the circle $\mathbb{T}$?

Solution

- (a) Suppose that there is a periodic solution to the ODE $x_t = f(x)$ satisfying $x(t+T) = x(t)$ for some $T > 0$ and all $t \in \mathbb{R}$. Using fundamental theorem of calculus, we find that

$$\int_0^T |x_t(t)|^2 dt = \int_0^T f(x) x_t(t) dt = \int_{x(0)}^{x(T)} f(s) ds = 0. \tag{2}$$

- However, notice that $|x_t|^2$ is a nonnegative continuous function on $[0, T]$. By above equality, we conlcude that $x_t(t) \equiv 0$ on $[0, T]$, which is saying that x must be a constant solution.

- (b) When the problem is posed on $\mathbb{T}$ and we use $\mathbb{R}$ as a covering space of $\mathbb{T}$ to describe the system, we are allowing solutions to have 2π-jumps within one period. As a consequence, the last equality of (2) could be nonzero.

5. (a) A Bernoulli equation is an ODE of the form

$$x_t = a(t)x + b(t)x^n$$

where a, b are continuous functions and $n \neq 1$. Show that the transformation

$$u = \frac{1}{x^{n-1}}$$

reduces a Bernoulli equation to a linear equation for u. Use this transformation to solve the logistic equation $x_t = x(1-x)$.

(b) A Riccati equation is an ODE of the form

$$x_t = a(t) + b(t)x + c(t)x^2.$$

where a, b, c are continuous functions, with $c \neq 0$. Show that the transformation

$$x = -\frac{u_t}{cu}$$

reduces the Riccati equation to a second order, linear equation for u. Use this transformation to solve the logistic equation $x_t = x(1-x)$.

Solution

- (a) If $u = 1/x^{n-1}$, we deduce that $u_t = (1-n)x_t/x^n$. Then the Bernoulli equation can be rewritten in terms of u as

$$u_t = (1-n)\left[a(t)u + b(t)\right],$$

  which is a linear equation for u.

- The logistic equation $x_t = x(1-x)$ is a special case of the Bernoulli equation with $a(t) = 1$, $b(t) = -1$, and $n = 2$. Using the transformation $u = 1/x$, we find that the equation is reduced to

$$u_t = -u + 1.$$

- We solve this ODE by separation of variables with initial condition $u(0) = 1/x_0$,

$$u(t) = \left(\frac{1}{x_0} - 1\right)e^{-t} + 1.$$

  Then the solution to the logistic equation is

$$x(t) = \frac{x_0}{(1-x_0)e^{-t} + x_0}.$$

- (b) If $x = -u_t/(cu)$, then we find that

$$-\frac{cuu_{tt} - cu_t^2 - c_t uu_t}{(cu)^2} = x_t = a(t) + b(t)x + c(t)x^2 = a - \frac{bu_t}{cu} + \frac{u_t^2}{cu^2}.$$

  Simplifying this equation yields

$$u_{tt} - \left(b + \frac{c_t}{c}\right) u_t + acu = 0.$$

- The logistic equation $x_t = x(1-x)$ is a special case of the Ricatti equation with $a(t) = 0$, $b(t) = 1$, and $c(t) = -1$. Using the transformation $x = u_t/u$, we find that the equation is reduced to

$$u_{tt} - u_t = 0.$$

- Solving this equation yields

$$u(t) = c_1 e^t + c_2.$$

  where c_1 and c_2 are constants. Notice that $x = \frac{d}{dt} \log |u|$. We obtain

$$x(t) = \frac{c_1}{c_1 + c_2 e^{-t}}.$$

  Assuming $x(0) = x_0$ gives the solution

$$x(t) = \frac{x_0}{(1 - x_0)e^{-t} + x_0}.$$