Problem set 4
Math 207A, Fall 2018
Due: Fri., Oct. 26

1. The ODE for a linear oscillator with displacement $x(t)$ is

$$\ddot{x} + \delta \dot{x} + \omega^2 x = 0,$$

where the damping coefficient $\delta \geq 0$ and the frequency $\omega > 0$ are constants. Write this ODE as a 2×2 first order linear system for (x, y) with $y = \dot{x}$. Sketch the phase plane for the following cases: (a) $\delta = 0$ (undamped); (b) $0 < \delta < 2\omega$ (underdamped); (c) $\delta = 2\omega$ (critically damped); (d) $\delta > 2\omega$ (overdamped). Classify the equilibrium $(x, y) = (0, 0)$ in each case. In which cases is the equilibrium hyperbolic?

2. (a) An $n \times n$ matrix N is said to be nilpotent if $N^k = 0$ for some $k \in \mathbb{N}$. Compute e^{tA} if $A = \lambda I + N$ where N is nilpotent. Justify all your steps.
(b) Compute e^{tA} if A is the 3×3 Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(c) Consider the 3×3 linear system $x_t = Ax$ where A is the matrix in (b) and $\lambda \in \mathbb{R}$. Describe the stable, unstable, and center subspaces and the stability of $x = 0$ in the cases (i) $\lambda < 0$; (ii) $\lambda = 0$; (iii) $\lambda > 0$.

3. Suppose that $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is a continuous, simultaneously diagonalizable, matrix valued function, meaning that there exists a constant matrix P and a diagonal matrix valued function $\Lambda : \mathbb{R} \to \mathbb{R}^{n \times n}$ with

$$\Lambda(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_n(t))$$

such that $A(t) = P\Lambda(t)P^{-1}$.
(a) Show that $A(s)$ and $A(t)$ commute for any $s, t \in \mathbb{R}$.
(b) Show that the solution of the initial value problem

$$x_t = A(t)x, \qquad x(0) = x_0$$

is given by

$$x(t) = e^{\int_0^t A(s)\, ds} x_0.$$

4. Let

$$A(t) = \begin{pmatrix} 1 & 2t \\ 0 & -1 \end{pmatrix}.$$

(a) Compute the fundamental matrix $\Phi(t; 0)$ that satisfies

$$\Phi_t = A(t)\Phi, \qquad \Phi(0; 0) = I.$$

(b) Compute

$$E(t) = e^{\int_0^t A(s)\, ds}$$

and show that $E(t) \neq \Phi(t; 0)$. Why doesn't the result of Problem 3 apply here?

5. Consider the Lorenz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z \end{aligned}$$

where $r, \beta, \sigma > 0$ are positive parameters. Determine the linearized stability of the equilibrium solution $(x, y, z) = (0, 0, 0)$. When is this equilibrium hyperbolic?