Problem set 4: Solutions
Math 207A, Fall 2018

1. The ODE for a linear oscillator with displacement $x(t)$ is

$$\ddot{x} + \delta\dot{x} + \omega^2 x = 0,$$

where the damping coefficient $\delta \geq 0$ and the frequency $\omega > 0$ are constants. Write this ODE as a 2×2 first order linear system for (x, y) with $y = \dot{x}$. Sketch the phase plane for the following cases: (a) $\delta = 0$ (undamped); (b) $0 < \delta < 2\omega$ (underdamped); (c) $\delta = 2\omega$ (critically damped); (d) $\delta > 2\omega$ (overdamped). Classify the equilibrium $(x, y) = (0, 0)$ in each case. In which cases is the equilibrium hyperbolic?

Solution

- By letting $y = \dot{x}$, the ODE can be written as

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x - \delta y, \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

  The only equilibrium of the system is $(x, y) = (0, 0)$.

- The eigenvalues of the coefficient matrix are

$$\lambda_\pm = \frac{-\delta \pm \sqrt{\delta^2 - 4\omega^2}}{2}.$$

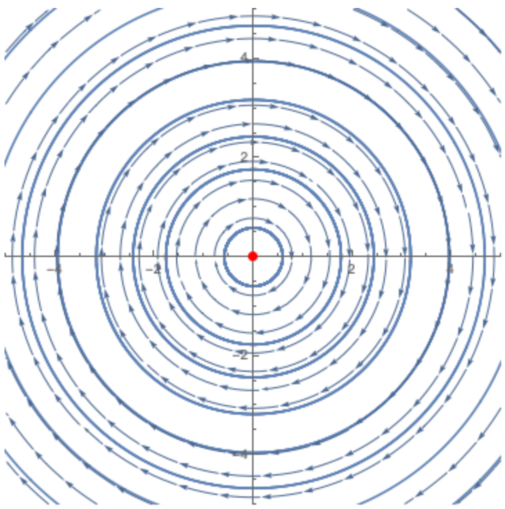
- (a) When the system is undamped ($\delta = 0$), we have $\lambda = \pm i\omega$, and the trajectories are limit cycles around a stable (but not asymptotically stable) center. The equilibrium is non-hyperbolic.

- (b) When the system is underdamped ($0 < \delta < 2\omega$), we have $\Re\lambda_\pm < 0$. It follows that the equilibrium is asymptotically stable, and the trajectories are spirals. The equilibrium is hyperbolic.

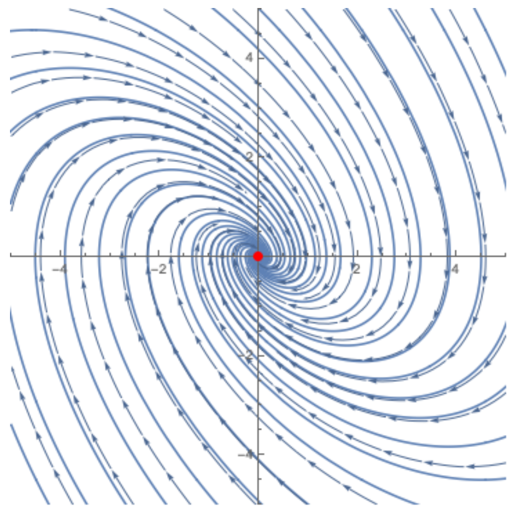
- (c) When the system is critically damped ($\delta = 2\omega$), we have $\lambda_\pm = -\delta/2$, and therefore, the equilibrium is asymptotically stable. The equilibrium is hyperbolic.

- (d) When the system is overdamped ($\delta > 2\omega$), we have $\lambda_\pm < 0$, and thus, the equilibrium is a stable node. The equilibrium is hyperbolic.

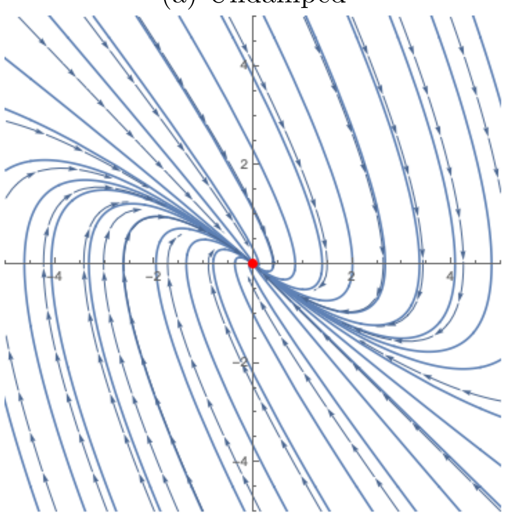
- The phase portraits are as follows



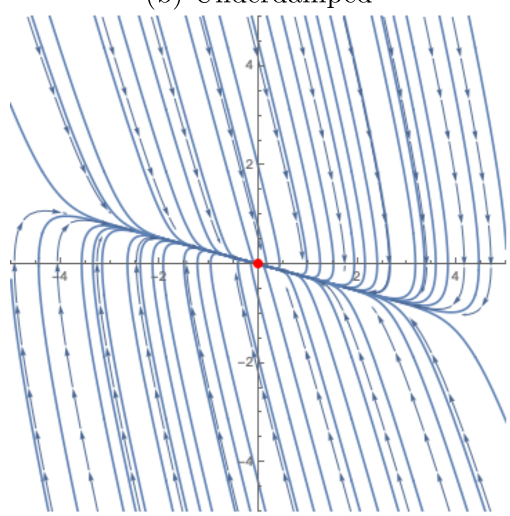
(a) Undamped



(b) Underdamped



(c) Critically damped



(d) Overdamped

2. (a) An $n \times n$ matrix N is said to be nilpotent if $N^k = 0$ for some $k \in \mathbb{N}$. Compute e^{tA} if $A = \lambda I + N$ where N is nilpotent. Justify all your steps.
(b) Compute e^{tA} if A is the 3×3 Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(c) Consider the 3×3 linear system $x_t = Ax$ where A is the matrix in (b) and $\lambda \in \mathbb{R}$. Describe the stable, unstable, and center subspaces and the stability of $x = 0$ in the cases (i) $\lambda < 0$; (ii) $\lambda = 0$; (iii) $\lambda > 0$.

Solution

- (a) Since identity matrix I commutes with any matrix, and also that $N^k = 0$, we can write

$$e^{tA} = e^{t\lambda I + tN} = e^{t\lambda} e^{tN} = e^{t\lambda} \sum_{j=0}^{\infty} \frac{t^j N^j}{j!} = e^{t\lambda} \sum_{j=0}^{k-1} \frac{t^j N^j}{j!}.$$

- (b) We can write $A = \lambda I + N$, where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad N^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k \geq 3.$$

  By part (a), we find that

$$e^{tA} = e^{t\lambda} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

- (c) Eigenvalues of A are λ with multiplicity 3. Eigenvector corresponding to λ is span $\{(1, 0, 0)\}$. However, the generalized eigenvectors corresponding λ span the whole space $\mathbb{R}^3$. If follows that the when (i) $\lambda < 0$, $E^s = \mathbb{R}^3$, and $E^c = E^u = \{0\}$; (i) $\lambda = 0$, $E^c = \mathbb{R}^3$, and $E^s = E^u = \{0\}$; (iii) $\lambda > 0$, $E^u = \mathbb{R}^3$, and $E^s = E^c = \{0\}$.

3. Suppose that $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuous, simultaneously diagonalizable, matrix valued function, meaning that there exists a constant matrix P and a diagonal matrix valued function $\Lambda : \mathbb{R} \to \mathbb{R}^{n\times n}$ with

$$\Lambda(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_n(t))$$

such that $A(t) = P\Lambda(t)P^{-1}$.
(a) Show that $A(s)$ and $A(t)$ commute for any $s, t \in \mathbb{R}$.
(b) Show that the solution of the initial value problem

$$x_t = A(t)x, \qquad x(0) = x_0$$

is given by

$$x(t) = e^{\int_0^t A(s)\,ds} x_0.$$

Solution

- (a) Since diagonal matrices commute, direct calculation shows that for any $s, t \in \mathbb{R}$

$$A(s)A(t) = P\Lambda(s)P^{-1}P\Lambda(t)P^{-1} = P\Lambda(t)P^{-1}P\Lambda(s)P^{-1} = A(t)A(s).$$

- (b) Since

$$\int_0^t A(s)ds = P\int_0^t \Lambda(s)dsP^{-1},$$

it follows that for any $j \in \mathbb{N}$

$$\left(\int_0^t A(s)ds\right)^j = P\left(\int_0^t \Lambda(s)ds\right)^j P^{-1}.$$

- Therefore, by definition of matrix exponential

$$e^{\int_0^t A(s)ds} = \sum_{j=0}^{\infty} \frac{1}{j!} P\left(\int_0^t \Lambda(s)ds\right)^j P^{-1}.$$

Using fundamental theorem of calculus, we obtain

$$\begin{aligned}
\frac{d}{dt} e^{\int_0^t A(s)\,ds} x_0 &= \sum_{j=1}^{\infty} \frac{1}{(j-1)!} P\Lambda(t)P^{-1}P\left(\int_0^t \Lambda(s)ds\right)^{j-1} P^{-1}x_0 \\
&= A(t)\sum_{j=0}^{\infty} \frac{1}{j!} P\Lambda(t)P^{-1}P\left(\int_0^t \Lambda(s)ds\right)^j P^{-1}x_0 \\
&= A(t)e^{\int_0^t A(s)\,ds} x_0.
\end{aligned}$$

4. Let

$$A(t) = \begin{pmatrix} 1 & 2t \\ 0 & -1 \end{pmatrix}.$$

(a) Compute the fundamental matrix $\Phi(t;0)$ that satisfies

$$\Phi_t = A(t)\Phi, \qquad \Phi(0;0) = I.$$

(b) Compute

$$E(t) = e^{\int_0^t A(s)\,ds}$$

and show that $E(t) \neq \Phi(t;0)$. Why doesn't the result of Problem 3 apply here?

Solution

- (a) The second equation of the system is essentially decoupled. The solution of $\dot{y} = -y$ with initial condition $y(0) = 1$ is $y(t) = e^{-t}$, where with initial condition $y(0) = 0$ is $y(t) = 0$.

- Considering the first equation of the system with, respectively, initial condition $x(0) = 1$ and $x(0) = 0$

$$\begin{cases} \dot{x} = x, \\ x(0) = 1, \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = x + 2te^{-t}, \\ x(0) = 0, \end{cases}$$

we find that the fundamental matrix of the system is

$$\Phi(t;0) = \begin{pmatrix} e^t & -te^{-t} + \sinh t \\ 0 & e^{-t} \end{pmatrix}.$$

- By induction, it is easy to see that

$$\left(\int_0^t A(s)ds \right)^j = \begin{pmatrix} t & t^2 \\ 0 & -t \end{pmatrix}^j = \begin{cases} t^j I & \text{if } j \text{ is even,} \\ t^{j-1} A(t) & \text{if } j \text{ is odd.} \end{cases}$$

Then it follows from definition of matrix exponential that

$$E(t) = \begin{pmatrix} e^t & t \sinh t \\ 0 & e^{-t} \end{pmatrix} \neq \Phi(t;0).$$

- The result of Problem 3 does not apply here because the eigenvectors of the matrix $A(t)$ depends on t, and thus, $A(t)$ and $A(s)$ do not commute for general $s, t \in \mathbb{R}$.

5. Consider the Lorenz equations

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z \end{aligned}$$

where $r, \beta, \sigma > 0$ are positive parameters. Determine the linearized stability of the equilibrium solution $(x, y, z) = (0, 0, 0)$. When is this equilibrium hyperbolic?

Solution

- The linearized system is

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y, \\ z_t &= -\beta z. \end{aligned}$$

- To determine linearized stability, we first solve for eigenvalues of the coefficient matrix
$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$
which are
$$\lambda_1 = -\beta, \qquad \lambda_{2,3} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}}{2}.$$
It is clear that $(\sigma + 1)^2 - 4\sigma(1 - r) > 0$ for all $r, \sigma > 0$, so that all the eigenvalues are real-valued.

- When $0 < r < 1$, we have $\lambda_{1,2,3} < 0$. Therefore, the equilibrium is linearized stable. Moreover, it is hyperbolic, which implies that the nonlinear system is asymptotically stable at this equilibrium.

- When $r = 1$, we have $\lambda_{1,3} < 0$, while $\lambda_2 = 0$. The linearized system is still linearized stable, but since the equilibrium is non-hyperbolic, we cannot say anything about the nonlinear stability.

- When $r > 1$, we have $\lambda_{1,3} < 0$ and $\lambda_2 > 0$. Then the system is linearly unstable. Since the equilibrium is hyperbolic, we conclude that the nonlinear system is also unstable in this circumstance.