Problem set 5

Math 207A, Fall 2018

Due: Fri., Nov. 2

1. A simple model for the potential energy of two uncharged molecules a distance r apart, with strong repulsion at small distances and weak attraction at large distances, is the Lennard-Jones potential

$$V(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right],$$

where $\epsilon, \sigma > 0$ are positive constants. Sketch the $(r, m\dot{r})$-phase plane for the motion of a particle of mass m and position $r(t) > 0$ in this potential. Sketch graphs of $r(t)$ versus t for various values of the energy of the particle.

2. The KPP or Fisher equation

$$u_t = u_{xx} + u(1-u)$$

is a PDE that describes the diffusion of a spatially distributed population with logistic growth. Traveling wave solutions $u = u(x - ct)$ satisfy the ODE

$$u'' + cu' + u(1-u) = 0.$$

Sketch the phase plane of this ODE for various values of the wave speed $c \geq 0$. For what values of c are there nonnegative, bounded traveling waves? Sketch the graph of $u(\xi)$ versus ξ for these values of c. What do these traveling waves describe?

3. Consider a linear system $x_t = A(t)x$ where the continuous matrix-valued function $A(t) = A(t+1)$ is 1-periodic, and $\Phi(t, t_0)$ is the fundamental matrix. Let $M = \Phi(1, 0)$ be the monodromy matrix and $L = \log M$ its logarithm. Show that there exists a 1-periodic matrix $\Psi(t) = \Psi(t+1)$ such that

$$\Phi(t, 0) = \Psi(t)e^{tL}.$$

Hint. You can assume that every nonsingular matrix M has a (possibly complex-valued) matrix logarithm $L = \log M$ such that $M = e^L$.

4. Consider the nonlinear system

$$x_t = -x + y + 3y^2, \qquad y_t = y.$$

(a) Sketch the phase plane, and show that its flow map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\varphi_t(x, y) = \left(xe^{-t} + y \sinh t + y^2(e^{2t} - e^{-t}), ye^t\right).$$

What are the stable and unstable manifolds of $(0, 0)$?

(b) Linearize the system at the equilibrium $(x, y) = (0, 0)$. Classify the equilibrium, sketch the phase plane of the linearized system, and show that its flow map $e^{tA} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$e^{tA} = \begin{pmatrix} e^{-t} & \sinh t \\ 0 & e^t \end{pmatrix}.$$

What are the stable and unstable subspaces?

(c) Show that the flow of the nonlinear system is mapped to the flow of the linearized system by $h : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$h(x, y) = \left(x - y^2, y\right).$$