Problem set 5: Solutions
Math 207A, Fall 2018

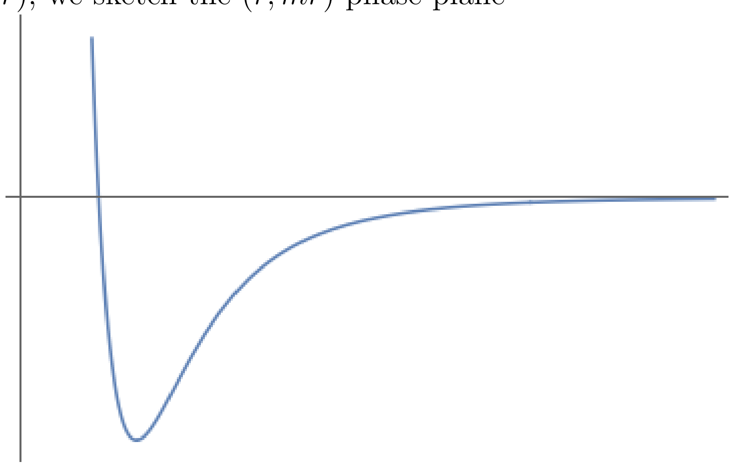
1. A simple model for the potential energy of two uncharged molecules a distance r apart, with strong repulsion at small distances and weak attraction at large distances, is the Lennard-Jones potential

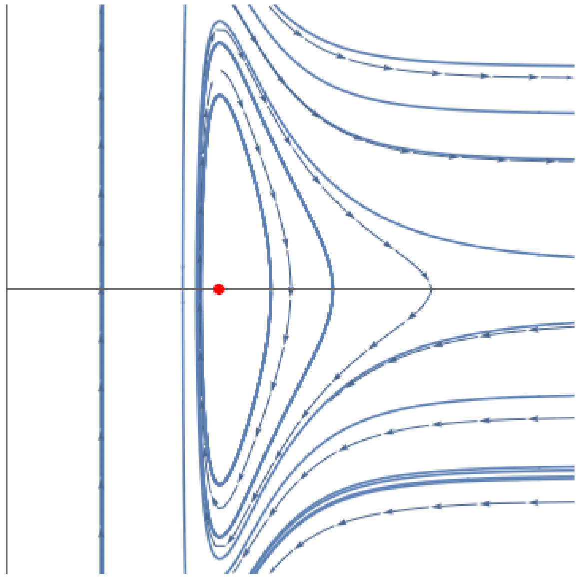
$$V(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right],$$

where $\epsilon, \sigma > 0$ are positive constants. Sketch the $(r, m\dot{r})$-phase plane for the motion of a particle of mass m and position $r(t) > 0$ in this potential. Sketch graphs of $r(t)$ versus t for various values of the energy of the particle.

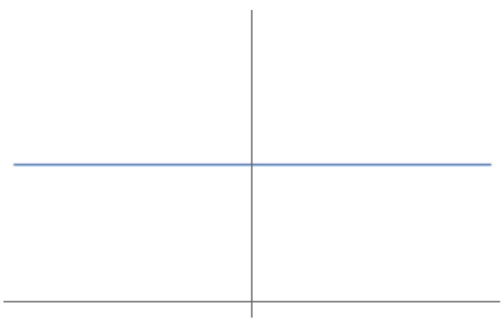
Solution

- The evolutionary equation of the system is the following gradient system
$$m\ddot{r} = -V'(r) = 24\epsilon\left(\frac{2\sigma^{12}}{r^{13}} - \frac{\sigma^6}{r^7}\right).$$
The only equilibrium for $r > 0$ is $r = \sqrt[6]{2}\sigma$.
- We can plot the profile of $V(r)$ for $r > 0$, and then based on the profile of $V(r)$, we sketch the $(r, m\dot{r})$-phase plane

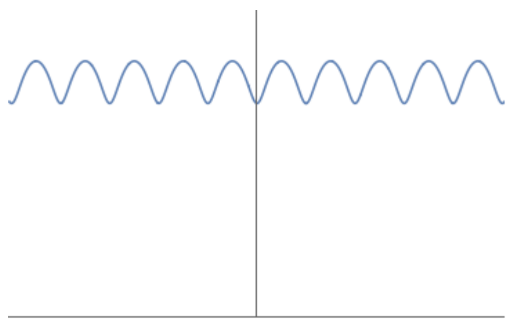




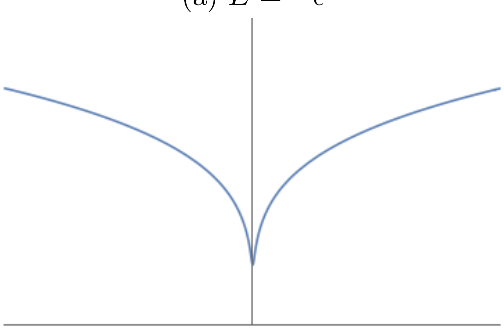
- Some possible profiles of $r(t)$ versus t for different values of the energy of the particle are



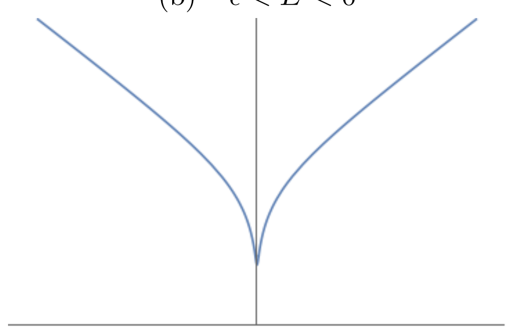
(a) $E = -\epsilon$



(b) $-\epsilon < E < 0$



(c) $E = 0$



(d) $E > 0$

2. The KPP or Fisher equation

$$u_t = u_{xx} + u(1-u)$$

is a PDE that describes the diffusion of a spatially distributed population with logistic growth. Traveling wave solutions $u = u(x - ct)$ satisfy the ODE

$$u'' + cu' + u(1-u) = 0.$$

Sketch the phase plane of this ODE for various values of the wave speed $c \geq 0$. For what values of c are there nonnegative, bounded traveling waves? Sketch the graph of $u(\xi)$ versus ξ for these values of c. What do these traveling waves describe?

Solution

- By letting $v = u'$, we can rewrite the ODE as

$$\begin{aligned} u' &= v, \\ v' &= -u(1-u) - cv. \end{aligned}$$

  The equilibria are $(u, v) = (0, 0)$ and $(u, v) = (1, 0)$.

- By linearizing the system around these two equilibria, we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{and} \quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

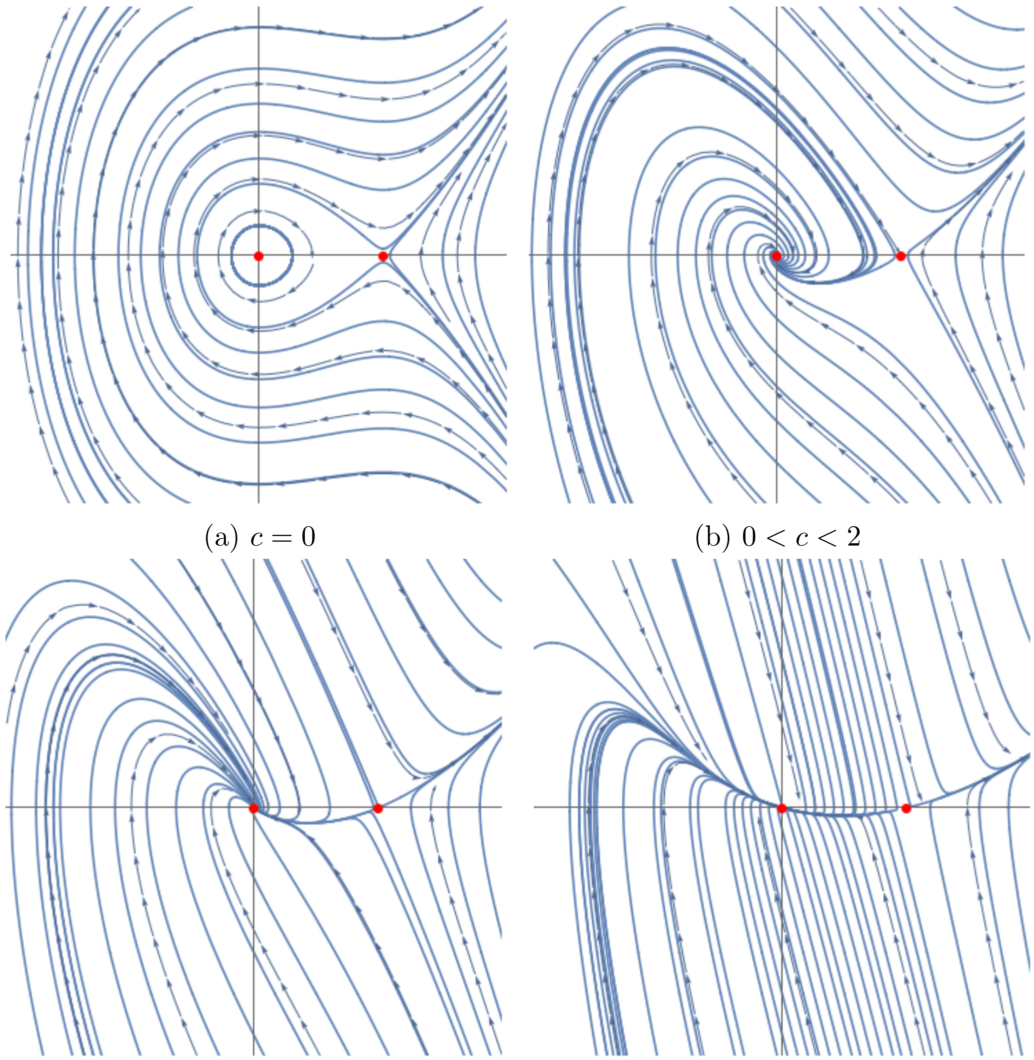
  We solve for eigenvalues of these coefficient matrices.

- At $(0, 0)$, eigenvalues are $\lambda_{1,2}^{(1)} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - 1}$.

  - When $c = 0$, $\lambda_{1,2}^{(1)} = \pm i$. This equilibrium is a center surrounded by closed periodic orbits.
  - When $0 < c < 2$, $\Re\lambda_{1,2}^{(2)} < 0$ and $\Im\lambda_{1,2}^{(2)} \neq 0$. This equilibrium is a stable spiral.
  - When $c \geq 2$, $\lambda_{1,2}^{(2)} < 0$. This equilibrium is a stable node.

- At $(1, 0)$, eigenvalues are $\lambda_{1,2}^{(2)} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1}$. It is clear that $\lambda_1^{(2)} > 0$ and $\lambda_2^{(2)} < 0$. Thus, this equilibrium is a saddle node.

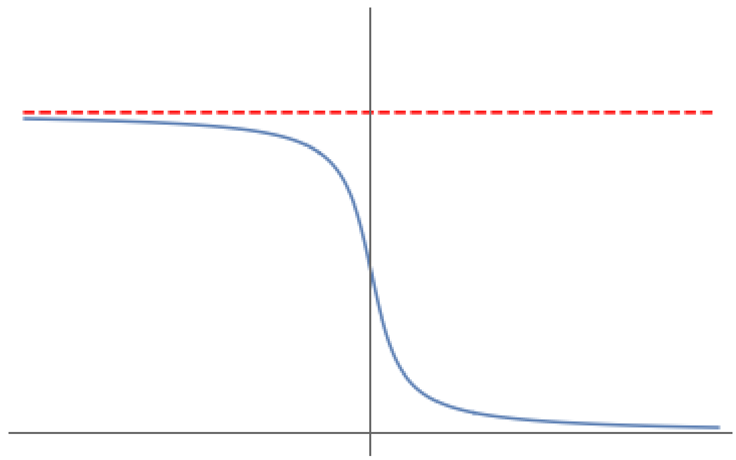
- Phase portraits of this ODE for various values of the waves speed $c \geq 0$ are



(a) $c = 0$ (b) $0 < c < 2$

(c) $c = 2$ (d) $2 < c$

- The only nonnegative, bounded traveling waves described in above phase planes are the heteroclinic orbits when $c \geq 2$. This orbit gives a solution $u(\xi)$ with profile



- These traveling waves describe the spread of a population from a fully populated region, where $u \to 1$ as $x \to -\infty$, into an unpopulated region, where $u \to 0$ as $x \to \infty$. Alternatively, in Fisher's original application, this solution describes the spread of a favorable gene from a population with the gene into a population without the gene.

3. Consider a linear system $x_t = A(t)x$ where the continuous matrix-valued function $A(t) = A(t+1)$ is 1-periodic, and $\Phi(t, t_0)$ is the fundamental matrix. Let $M = \Phi(1, 0)$ be the monodromy matrix and $L = \log M$ its logarithm. Show that there exists a 1-periodic matrix $\Psi(t) = \Psi(t + 1)$ such that

$$\Phi(t, 0) = \Psi(t)e^{tL}.$$

HINT. You can assume that every nonsingular matrix M has a (possibly complex-valued) matrix logarithm $L = \log M$ such that $M = e^L$.

Solution

- Since $\Phi(t, t_0)$ is the fundamental matrix, and that $A(t + 1) = A(t)$, it is clear that

$$\frac{d}{dt}\Phi(t + 1, 1) = A(t + 1)\Phi(t + 1, 1) = A(t)\Phi(t + 1, 1).$$

  Also notice that $\Phi(1, 1) = I$, the identity matrix. Since also $\Phi(t, 0)$ satisfies

$$\begin{aligned} &\frac{d}{dt}\Phi(t, 0) = A(t)\Phi(t, 0), \\ &\Phi(0, 0) = I, \end{aligned}$$

  then by Picard-Lindelof theorem, we have $\Phi(t + 1, 1) = \Phi(t, 0)$.

- Let $M = \Phi(1, 0)$ be the monodromy matrix, which is clearly nonsingular, and thus, $L = \log M$ exists. We also denote $\Psi(t) = \Phi(t, 0)e^{-tL}$. It follows that

$$\begin{aligned} \Psi(t + 1) &= \Phi(t + 1, 0)e^{-L}e^{-tL} \\ &= \Phi(t + 1, 1)\Phi(1, 0)M^{-1}e^{-tL} \\ &= \Phi(t, 0)e^{-tL} \\ &= \Psi(t), \end{aligned}$$

  which show that $\Psi(t)$ is a 1-periodic matrix.

4. Consider the nonlinear system

$$x_t = -x + y + 3y^2, \qquad y_t = y.$$

(a) Sketch the phase plane, and show that its flow map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\varphi_t(x, y) = \left(xe^{-t} + y \sinh t + y^2(e^{2t} - e^{-t}), ye^t\right).$$

What are the stable and unstable manifolds of $(0, 0)$?

(b) Linearize the system at the equilibrium $(x, y) = (0, 0)$. Classify the equilibrium, sketch the phase plane of the linearized system, and show that its flow map $e^{tA} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$e^{tA} = \begin{pmatrix} e^{-t} & \sinh t \\ 0 & e^t \end{pmatrix}.$$

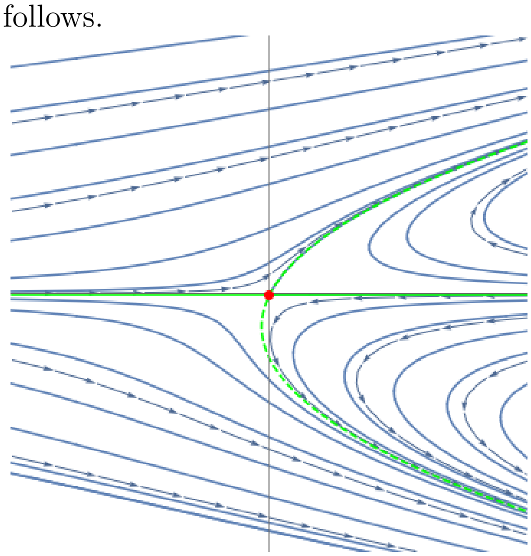
What are the stable and unstable subspaces?

(c) Show that the flow of the nonlinear system is mapped to the flow of the linearized system by $h : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$h(x, y) = \left(x - y^2, y\right).$$

Solution

- (a) The only equilibrium is $(x, y) = (0, 0)$. The phase portrait of the system is as follows.



- The second equation $y_t = y$ is decoupled from the system. Solving this equation with initial condition $y(0) = y_0$ gives

$$y(t) = y_0 e^t.$$

Substituting this into the first equation with initial condition $x(0) = x_0$, and by using integrating factors, we can find that the solution is

$$x(t) = x_0 e^{-t} + y_0 \sinh t + y_0^2(e^{2t} - e^{-t}),$$

which show that $\varphi_t(x, y)$ defines the flow map.

- To find the stable manifold of $(0, 0)$, notice that $\lim_{t\to\infty} e^t = \infty$, we must have $y = 0$ by looking at the second component of the flow map. Since $\lim_{t\to\infty} e^{-t} = 0$, x can be any point on $\mathbb{R}$. Therefore, the stable manifold is

$$W^{(s)}(0, 0) = \{(x, 0) \mid x \in \mathbb{R}\}.$$

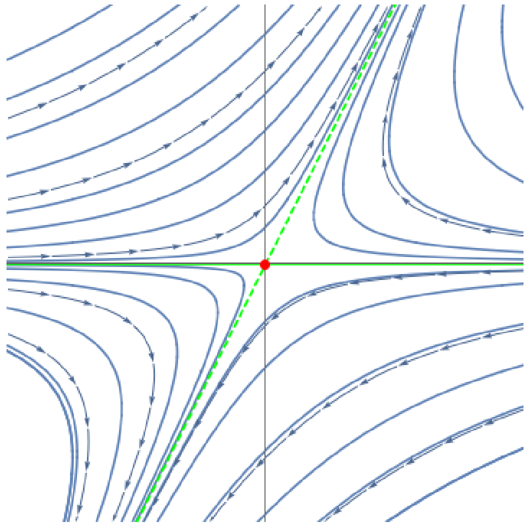
- Similarly, to find unstable manifold, we must have the coefficients of e^{-t} add to zero. Therefore,

$$W^{(u)}(0, 0) = \left\{(x, y) \mid x - \frac{y}{2} - y^2 = 0\right\}.$$

- (b) The linearized system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Eigenvalues and corresponding eigenvectors are $\lambda_1 = -1$, $r_1 = (1, 0)$ and $\lambda_2 = 1$, $r_2 = (1, 2)$. The phase portrait is as follows, which is qualitatively the same as the phase portrait of the nonlinear system (the equilibrium is hyperbolic).



- The stable subspace of $(0,0)$ is $E^{(s)}(0,0) = \operatorname{span}\{(1,0)\}$, while the unstable subspace is $E^{(u)}(0,0) = \operatorname{span}\{(1,2)\}$.

- (c) To show the topological conjugacy between the linear and the non-linear system, it suffices to show

$$h \circ \varphi_t = e^{tA} \circ h.$$

Indeed, direct computation shows that

$$\begin{aligned} h \circ \varphi_t(x,y) &= \begin{pmatrix} xe^{-t} + y\sinh t + y^2e^{2t} - y^2e^{-t} - y^2e^{2t} \\ ye^t \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}(x-y^2) + y\sinh t \\ ye^t \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & \sinh t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x - y^2 \\ y \end{pmatrix} \\ &= e^{tA}h(x,y). \end{aligned}$$